

# Computations in Modules over Commutative Domains

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## Часть I

# System of Linear Equations

Let  $R$  be a commutative domain,  $F$  be the field of fractions of  $R$ .  
 $A \in R^{n \times m}$ ,  $c \in R^n$ ,  $n \leq m$ ,  $A^* = (A, c) = (a_{ij})$  and,

$$Ax = c$$

be a system of linear equations over  $R$

$$x_i = (\delta_{i,m+1}^n - \sum_{j=n+1}^m x_j \delta_{ij}^n) (\delta^n)^{-1}, \quad i = 1 \dots n,$$

where  $x_j, j = n + 1, \dots, m$ , are the free variables, and the determinant  $\delta^n \neq 0$ .

We denote by  $\delta^k$ ,  $k = 1, \dots, n$  the left upper corner minor of matrix  $A$  of order  $k$ , and by  $\delta_{ij}^k$  the corner minor of matrix  $A$  where columns  $i$  and  $j$  have been interchanged. We assume that all corner minors  $\delta^k$ ,  $k = 1, \dots, n$  are different from 0.

- 1 Dodgson's Algorithm
- 2 Method of Forward and Backward Direction
- 3 The One-pass Method
- 4 The Recursive Method

## The determinant identity

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & k \end{vmatrix} = \begin{vmatrix} \begin{vmatrix} a & b \\ d & e \end{vmatrix} & \begin{vmatrix} b & c \\ e & f \end{vmatrix} \\ \begin{vmatrix} d & e \\ g & h \end{vmatrix} & \begin{vmatrix} e & f \\ h & k \end{vmatrix} \end{vmatrix} \cdot e^{-1}$$

or in the more general form

$$\hat{a}_{ij}^{k+1} = \begin{vmatrix} \hat{a}_{i-1,j-1}^k & \hat{a}_{i-1,j}^k \\ \hat{a}_{i,j-1}^k & \hat{a}_{i,j}^k \end{vmatrix} \cdot (\hat{a}_{i-1,j-1}^{k-1})^{-1}$$

where

$$\hat{a}_{ij}^{k+1} = |A_{j-k, \dots, j: (columns)}^{i-k, \dots, i: (rows)}|$$

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$$a_{ij}^{k+1} = (a_{kk}^k a_{ij}^k - a_{ik}^k a_{kj}^k)(a_{k-1,k-1}^{k-1})^{-1},$$

$$k = 2, \dots, n-1, \quad i = k+1, \dots, n, \quad j = k+1, \dots, m,$$

where

$$a_{ij}^{k+1} = |A_{1, \dots, k, j: (\text{columns})}^{1, \dots, k, i: (\text{rows})}|$$

$$\begin{pmatrix} a_{1,1}^1 & a_{1,2}^1 & \cdots & a_{1,n-1}^1 & a_{1,n}^1 & a_{1,n}^1 & \cdots & a_{1,m+1}^1 \\ 0 & a_{2,2}^2 & \cdots & a_{2,n-1}^2 & a_{2,n}^2 & a_{2,n+1}^2 & \cdots & a_{2,m+1}^2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{n-1,n-1}^{n-1} & a_{n-1,n}^{n-1} & a_{n-1,n+1}^{n-1} & \cdots & a_{n-1,m+1}^{n-1} \\ 0 & 0 & \cdots & 0 & a_{n,n}^n & a_{n,n+1}^n & \cdots & a_{n,m+1}^n \end{pmatrix}$$

$$a_{k,k}^k \neq 0, k = 1, \dots, n-1$$

$\delta_{ij}^n$ .  $\delta_{ij}^k$  is the corner minor of order  $k$  of the matrix  $A$  after column  $i$  has been interchanged with column  $j$ . The determinant identity of the backward direction algorithm is:

$$\delta_{ij}^n = (a_{nn}^n a_{ij}^i - \sum_{k=i+1}^n a_{ik}^i \delta_{kj}^n) (a_{ii}^i)^{-1}, \quad i = n-1, \dots, 1, \quad j = n+1, \dots, m.$$

$$\begin{pmatrix} a_{n,n}^n & 0 & \cdots & 0 & 0 & \delta_{1,n}^n & \cdots & \delta_{1,m+1}^n \\ 0 & a_{n,n}^n & \cdots & 0 & 0 & \delta_{2,n+1}^n & \cdots & \delta_{2,m+1}^n \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{n,n}^n & 0 & \delta_{n-1,n+1}^n & \cdots & \delta_{n-1,m+1}^n \\ 0 & 0 & \cdots & 0 & a_{n,n}^n & \delta_{n,n+1}^n & \cdots & \delta_{n,m+1}^n \end{pmatrix}$$

$$\begin{aligned} & \begin{pmatrix} 3 & 1 & 1 & -1 & 4 \\ 1 & 2 & 0 & 1 & 4 \\ 0 & 1 & 2 & 0 & -2 \\ 1 & 0 & 0 & 2 & -1 \end{pmatrix} \rightarrow \\ \rightarrow & \begin{pmatrix} 3 & 1 & 1 & -1 & 4 \\ 0 & 5 & -1 & 4 & 8 \\ 0 & 3 & 6 & 0 & -6 \\ 0 & -1 & -1 & 7 & -7 \end{pmatrix} \rightarrow \begin{pmatrix} 3 & 1 & 1 & -1 & 4 \\ 0 & 5 & -1 & 4 & 8 \\ 0 & 0 & 11 & -4 & -18 \\ 0 & 0 & -2 & 13 & -9 \end{pmatrix} \rightarrow \\ \rightarrow & \begin{pmatrix} 3 & 1 & 1 & -1 & 4 \\ 0 & 5 & -1 & 4 & 8 \\ 0 & 0 & 11 & -4 & -18 \\ 0 & 0 & 0 & 27 & -27 \end{pmatrix} \rightarrow \begin{pmatrix} 3 & 1 & 1 & -1 & 4 \\ 0 & 5 & -1 & 4 & 8 \\ 0 & 0 & 27 & 0 & -54 \\ 0 & 0 & 0 & 27 & -27 \end{pmatrix} \rightarrow \\ \rightarrow & \begin{pmatrix} 3 & 1 & 1 & -1 & 4 \\ 0 & 27 & 0 & 0 & 54 \\ 0 & 0 & 27 & 0 & -54 \end{pmatrix} \rightarrow \begin{pmatrix} 27 & 0 & 0 & 0 & 27 \\ 0 & 27 & 0 & 0 & 54 \\ 0 & 0 & 27 & 0 & -54 \end{pmatrix} \end{aligned}$$

The number of operations, necessary for the procedure of forward and backward direction, is

$$N^m = (9n^2m - 5n^3 - 3nm - 3n^2 - 6m + 8n)/6,$$

$$N^d = (3n^2m - n^3 - 3nm - 6n^2 + 13n - 6)/6$$

$$N^a = (6n^2m - 4n^3 - 6nm + 3n^2 + n)/6.$$

- 1 Dodgson's Algorithm
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$$\delta_{k+1,j}^{k+1} = a_{k+1,k+1}\delta_{kk}^k - \sum_{p=1}^k a_{k+1,p}\delta_{pj}^k, j = k + 1 \dots m,$$

$$\delta_{ij}^{k+1} = (\delta_{k+1,k+1}^{k+1}\delta_{i,j}^k - \delta_{k+1,j}^{k+1}\delta_{i,k+1}^k) / \delta_{k,k}^k,$$

$$k = 1, \dots, n - 1, i = 1, \dots, k, j = k + 2, \dots, m.$$

$$\begin{pmatrix} a_{k,k}^k & 0 & \cdots & 0 & \delta_{1,k+1}^k & \cdots & \delta_{1,m+1}^k \\ 0 & a_{k,k}^k & \cdots & 0 & \delta_{2,k+1}^k & \cdots & \delta_{2,m+1}^k \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{k,k}^k & \delta_{k,k+1}^k & \cdots & \delta_{k,m+1}^k \\ a_{k+1,1} & a_{k+1,2} & \cdots & a_{k+1,k} & a_{k+1,k+1} & \cdots & a_{k+1,m+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,k} & a_{n,k+1} & \cdots & a_{n,m+1} \end{pmatrix}$$

$$\begin{aligned}
 & \begin{pmatrix} 3 & 1 & 1 & -1 & 4 \\ 1 & 2 & 0 & 1 & 4 \\ 0 & 1 & 2 & 0 & -2 \\ 1 & 0 & 0 & 2 & -1 \end{pmatrix} \rightarrow \\
 \rightarrow & \begin{pmatrix} 3 & 1 & 1 & -1 & 4 \\ 0 & 5 & -1 & 4 & 8 \\ 0 & 1 & 2 & 0 & -2 \\ 1 & 0 & 0 & 2 & -1 \end{pmatrix} \rightarrow \begin{pmatrix} 5 & 0 & 2 & -3 & 4 \\ 0 & 5 & -1 & 4 & 8 \\ 0 & 1 & 2 & 0 & -2 \\ 1 & 0 & 0 & 2 & -1 \end{pmatrix} \rightarrow \\
 \rightarrow & \begin{pmatrix} 5 & 0 & 2 & -3 & 4 \\ 0 & 5 & -1 & 4 & 8 \\ 0 & 0 & 11 & -4 & -18 \\ 1 & 0 & 0 & 2 & -1 \end{pmatrix} \rightarrow \begin{pmatrix} 11 & 0 & 0 & -5 & 16 \\ 0 & 11 & 0 & 8 & 14 \\ 0 & 0 & 11 & -4 & -18 \\ 1 & 0 & 0 & 2 & -1 \end{pmatrix} \rightarrow \\
 \rightarrow & \begin{pmatrix} 11 & 0 & 0 & -5 & 16 \\ 0 & 11 & 0 & 8 & 14 \\ 0 & 0 & 11 & -4 & -18 \end{pmatrix} \rightarrow \begin{pmatrix} 27 & 0 & 0 & 0 & 27 \\ 0 & 27 & 0 & 0 & 54 \\ 0 & 0 & 27 & 0 & -54 \end{pmatrix}
 \end{aligned}$$



$$N^m = (9n^2m - 6n^3 - 3nm - 6m + 6n)/6,$$

$$N^d = (3n^2m - 2n^3 - 3nm - 6m + 2n + 12)/6$$

$$N^a = (6n^2m - 4n^3 - 6nm + 3n^2 + n)/6.$$

Number of Operations			
Method	Multiplications	Divisions	Add./Substr.
FB	$\frac{(4n^3+3n^2-n-6)}{6}$	$\frac{(2n^3-6n^2+10n-6)}{6}$	$\frac{(2n^3+3n^2-5n)}{6}$
OP	$\frac{(n^3+2n^2-n-2)}{2}$	$\frac{(n^3-7n+6)}{6}$	$\frac{(2n^3+3n^2-5n)}{6}$

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$$A_{k,c}^{r,l,(p)} = \begin{pmatrix} a_{r+1,k+1}^p & a_{r+1,k+2}^p & \cdots & a_{r+1,c}^p \\ a_{r+2,k+1}^p & a_{r+2,k+2}^p & \cdots & a_{r+2,c}^p \\ \vdots & \vdots & \ddots & \vdots \\ a_{l,k+1}^p & a_{l,k+2}^p & \cdots & a_{l,c}^p \end{pmatrix},$$

$$G_{k,c}^{r,l,(p)} = \begin{pmatrix} \delta_{r+1,k+1}^p & \delta_{r+1,k+2}^p & \cdots & \delta_{r+1,c}^p \\ \delta_{r+2,k+1}^p & \delta_{r+2,k+2}^p & \cdots & \delta_{r+2,c}^p \\ \vdots & \vdots & \ddots & \vdots \\ \delta_{l,k+1}^p & \delta_{l,k+2}^p & \cdots & \delta_{l,c}^p \end{pmatrix},$$

$G_{k,c}^{r,l,(p)}, A_{k,c}^{r,l,(p)} \in \mathbf{R}^{(l-r) \times (c-k)}, 0 \leq k < n, k < c \leq n, 0 \leq r < m, r < l \leq m, 1 \leq p \leq n.$

We describe one recursive step reducing the matrix  $\tilde{A} = A_{k,c}^{k,l,(k+1)}$  to the diagonal form

$$\tilde{A} \rightarrow (\delta^l I_{l-k}, \hat{G})$$

where

$$\tilde{A} = A_{k,c}^{k,l,(k+1)}, \quad \hat{G} = G_{l,c}^{k,l,(l)}$$

$0 \leq k < c \leq m, k < l \leq n, l < c.$  Note that if  $k = 0, l = n$  and  $c = m,$  then we obtain the solution of the original system.

## Description of One Step of the Recursive Method

$$\begin{aligned}
 \tilde{A} = \begin{pmatrix} A^1 \\ A^2 \end{pmatrix} &\rightarrow_1 \begin{pmatrix} \delta^s I_{s-k} & G_2^1 \\ A_1^2 & A_2^2 \end{pmatrix} \rightarrow_2 \begin{pmatrix} \delta^s I_{s-k} & G_2^1 \\ 0 & \hat{A}_2^2 \end{pmatrix} \rightarrow_3 \\
 &\rightarrow_3 \begin{pmatrix} \delta^s I_{s-k} & G_{2'}^1 & G_{2''}^1 \\ 0 & \delta^l I_{l-s} & \hat{G}_{2''}^2 \end{pmatrix} \rightarrow_4 \begin{pmatrix} \delta^l I_{s-k} & 0 & \hat{G}_{2''}^1 \\ 0 & \delta^l I_{l-s} & \hat{G}_{2''}^2 \end{pmatrix} = \\
 &= \begin{pmatrix} \delta^l I_{l-k} & \hat{G} \end{pmatrix}
 \end{aligned}$$

$s: k < s < l \tilde{A}$ :

$$\tilde{A} = \begin{pmatrix} A^1 \\ A^2 \end{pmatrix},$$

where  $A^1 = A_{k,c}^{k,s,(k+1)}$  is the upper part of the matrix  $\tilde{A}$  consisting of  $s - k$  rows and  $A^2 = A_{k,c}^{s,l,(k+1)}$  is the lower part of the matrix  $\tilde{A}$ .

$$A^1 \rightarrow (\delta^s I_{s-k}, G_2^1), \quad (1)$$

where  $A^1 \in \mathbf{R}^{(s-k) \times (c-k)}$ ,  $G_2^1 = G_{s,c}^{k,s,(s)}$ .

Let  $A^2 = (A_1^2, A_2^2)$  where  $A_1^2 = A_{k,s}^{s,l,(k+1)}$  and  $A_2^2 = A_{s,c}^{s,l,(k+1)}$  consisting of  $s - k$  and  $c - s$  columns respectively,  $\delta^k \neq 0$ . The matrix  $\hat{A}_2^2 = A_{s,c}^{s,l,(s+1)}$  is computed with the help of the matrix identity

$$\hat{A}_2^2 = (\delta^s \cdot A_2^2 - A_1^2 \cdot G_2^1)(\delta^k)^{-1}. \quad (II)$$

$$\hat{A}_2^2 \rightarrow (\delta^l I_{l-s}, \hat{G}_{2''}^2), \quad (III)$$

where  $\hat{A}_2^2 \in \mathbf{R}^{(l-s) \times (c-s)}$  and  $\hat{G}_{2''}^2 = G_{l,c}^{s,l,(l)}$ .

Let  $G_2^1 = (G_{2'}^1, G_{2''}^1)$ , where the blocks  $G_{2'}^1 = G_{s,l}^{k,s,(s)}$  and  $G_{2''}^1 = G_{l,c}^{k,s,(s)}$  contain  $l - s$  and  $c - l$  columns respectively, and  $\delta^s \neq 0$ .

The matrix  $\hat{G}_{2''}^1 = G_{l,c}^{k,s,(l)}$  is computed with the help of the matrix identity

$$\hat{G}_{2''}^1 = (\delta^l \cdot G_{2''}^1 - G_{2'}^1 \cdot \hat{G}_{2''}^2)(\delta^s)^{-1}. \quad (IV)$$

In the result we obtain  $\delta^l$  and

$$\hat{G} = \begin{pmatrix} \hat{G}_{2''}^1 \\ \hat{G}_{2''}^2 \end{pmatrix}$$



Example:

$$\begin{pmatrix} 3 & 1 & 1 & -1 \\ 1 & 2 & 0 & 1 \\ 0 & 1 & 2 & 0 \\ 1 & 0 & 0 & 2 \end{pmatrix}$$

$$\cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \\ -2 \\ -1 \end{pmatrix} A^1 = A_{05}^{02(1)} \rightarrow (\delta^2 I_2, G_{25}^{02(2)}) \quad (I)$$

$$A_{05}^{01(1)} \rightarrow (\delta^1 I_1, G_{15}^{01(1)}) = (3; 1, 1, -1, 4) \quad (i)$$

$$\delta^0 A_{15}^{12(2)} = \delta^1 A_{15}^{12(1)} - A_{01}^{12(1)} G_{15}^{01(1)} =$$

$$= 3(2, 0, 1, 4) - (1)(1, 1, -1, 4) = (5, -1, 4, 8), \quad \delta^0 \equiv 1. \quad (ii)$$

$$A_{15}^{12(2)} \rightarrow (\delta^2 h_1, G_{25}^{12(2)}) = (5; -1, 4, 8) \quad (iii)$$

$$\delta^1 G_{25}^{01(2)} = \delta^2 G_{25}^{01(1)} - G_{12}^{01(1)} G_{25}^{12(2)} =$$

$$= 5(1, -1, 4) - (1)(-1, 4, 8) = (6, -9, 12) \quad (iv)$$

$$G_{25}^{01(2)} = (2, -3, 4)$$

$$(\delta^2 I_2, G_{25}^{02(2)}) = \begin{pmatrix} 5 & 0; & 2 & -3 & 4 \\ 0 & 5; & -1 & 4 & 8 \end{pmatrix}$$

$$\hat{A}_2^2 = A_{25}^{24(3)} \quad (II)$$

$$\delta^0 A_{25}^{24(3)} = \delta^2 A_{25}^{24(1)} - A_{02}^{24(1)} G_{25}^{02(2)} =$$

$$= 5 \cdot \begin{pmatrix} 2 & 0 & -2 \\ 0 & 2 & -1 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & -3 & 4 \\ -1 & 4 & 8 \end{pmatrix} =$$

$$= \begin{pmatrix} 11 & -4 & -18 \\ -2 & 13 & -9 \end{pmatrix}$$

$$\delta^0 \equiv 1; \quad A_{25}^{24(3)} = \begin{pmatrix} 11 & -4 & -18 \\ -2 & 13 & -9 \end{pmatrix}$$

$$A_{25}^{24(3)} \rightarrow (\delta^4 I_2, G_{45}^{24(4)}) \quad (III)$$

$$A_{25}^{23(3)} \rightarrow (\delta^3 I_1, G_{35}^{23(3)}) = (11; -4, -18) \quad (i)$$

$$\begin{aligned} \delta^2 A_{35}^{34(4)} &= \delta^3 A_{35}^{34(4)} - A_{23}^{34(3)} G_{35}^{23(3)} = \\ &= 11(13, -9) - (-2)(-4, -18) = (135, -135) \end{aligned}$$

$$A_{35}^{34(4)} = (27, -27) \quad (ii)$$

$$A_{35}^{34(4)} \rightarrow (\delta^4 I_1, G_{45}^{34(4)}) = (27, -27) \quad (iii)$$

$$\begin{aligned} \delta^3 G_{45}^{23(4)} &= \delta^4 G_{45}^{23(3)} - G_{34}^{23(3)} G_{45}^{34(4)} = \\ &= 27(-18) - (-4)(-27) = -594, \quad G_{45}^{23(4)} = (-54) \quad (iv) \end{aligned}$$

$$(\delta^4 I_2, G_{45}^{24(4)}) = \begin{pmatrix} 27 & 0; & -54 \\ 0 & 27; & -27 \end{pmatrix}$$

$$\hat{G}_{2''}^1 = G_{45}^{02(4)} \quad (V)$$

$$\begin{aligned}
 \delta^2 G_{45}^{02(4)} &= \delta^4 G_{45}^{02(2)} - G_{24}^{02(2)} G_{45}^{24(4)} = \\
 &= 27 \begin{pmatrix} 4 \\ 8 \end{pmatrix} - \begin{pmatrix} 2 & -3 \\ -1 & 4 \end{pmatrix} \begin{pmatrix} -54 \\ -27 \end{pmatrix} = \begin{pmatrix} 135 \\ 270 \end{pmatrix} \\
 G_{45}^{02(4)} &= \begin{pmatrix} 27 \\ 54 \end{pmatrix} \\
 \delta^4 &= 27; \quad G_{45}^{04(4)} = \begin{pmatrix} 27 \\ 54 \\ -54 \\ -27 \end{pmatrix}
 \end{aligned}$$

## Example: Description of the First Step

$$A = \begin{pmatrix} 3 & 1 & 1 & -1 & 4 \\ 1 & 2 & 0 & 1 & 4 \\ 0 & 1 & 2 & 0 & -2 \\ 1 & 0 & 0 & 2 & -1 \end{pmatrix} A \rightarrow_1$$

$$\begin{pmatrix} 5 & 0 & 2 & -3 & 4 \\ 0 & 5 & -1 & 4 & 8 \\ 0 & 1 & 2 & 0 & -2 \\ 1 & 0 & 0 & 2 & -1 \end{pmatrix} \rightarrow_2 \begin{pmatrix} 5 & 0 & 2 & -3 & 4 \\ 0 & 5 & -1 & 4 & 8 \\ 0 & 0 & 11 & -4 & -18 \\ 0 & 0 & -2 & 13 & -9 \end{pmatrix} \rightarrow_3$$

$$\rightarrow_3 \begin{pmatrix} 5 & 0 & 2 & -3 & 4 \\ 0 & 5 & -1 & 4 & 8 \\ 0 & 0 & 27 & 0 & -54 \\ 0 & 0 & 0 & 27 & -27 \end{pmatrix} \rightarrow_4 \begin{pmatrix} 27 & 0 & 0 & 0 & 27 \\ 0 & 27 & 0 & 0 & 54 \\ 0 & 0 & 27 & 0 & -54 \\ 0 & 0 & 0 & 27 & -27 \end{pmatrix}$$

**Complexity of the Recursive Method is  $O(mn^{\beta-1})$ .**

We can obtain an exact estimate. For  $n = 2^N$ ,  $m = n + 1$  and  $\beta = \log_2 7$  the number of multiplication operations is

$$\frac{7}{15}n^{\log_2 7} + n^2(\log_2 n - \frac{2}{3}) + n(2\log_2 n + \frac{1}{5}).$$

For  $n = 2^N$ ,  $\beta = 3$  the number of multiplications and divisions is

$$N^m = (6n^2m - 4n^3 + (6nm - 3n^2)\log_2 n - 6nm + 4n)/6,$$

$$N^d = ((6nm - 3n^2)\log_2 n - 6nm - n^2 + 6m + 3n - 2)/6.$$

The number of multiplication operations for  $m = n + 1$  is  $(1/3)n^3 + O(n^2)$ . The estimations for the previous two methods are, respectively,  $n^3 + O(n^2)$  and  $(2/3)n^3 + O(n^2)$ .



## Часть II

# Adjoint Matrix

The best method for computing the matrix determinant and adjoint matrix in the arbitrary commutative ring was suggested in the papers by Kaltofen [9] and Kaltofen and Villard [10]. Its complexity is  $O(n^{\beta+1/3} \log n \log \log n)$ .

Let  $\mathcal{A} = \begin{pmatrix} A & C \\ B & D \end{pmatrix}$  be an invertible matrix and  $A$  an invertible block.

Then

$$\mathcal{A}^{-1} = \begin{pmatrix} I & -A^{-1}C \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & (D - BA^{-1}C)^{-1} \end{pmatrix} \begin{pmatrix} I & 0 \\ -B & I \end{pmatrix} \begin{pmatrix} A^{-1} \\ 0 \end{pmatrix}$$

In case  $n = 2^p$  it will take  $2^{p-1}$  inversions of  $2 \times 2$  blocks and  $2^{p-k}$  multiplications of  $2^k \times 2^k$  blocks.

$$n^{\log 7} - n/2$$

$$O(n^\beta)$$

$$O(n^\beta)$$

Let  $R$  be a commutative ring, and let  $\mathcal{A} = (a_{i,j})$  be a square matrix of order  $n$  over the ring  $R$ .

$$\mathcal{A}_t^{(s)} = (a_{i,j}^s)_{j=s,\dots,t}^{i=s,\dots,t} \quad \text{and} \quad \mathcal{G}_s^{(t)} = (\delta_{t(i,j)})_{j=t+1,\dots,n}^{i=s,\dots,t}$$

## Theorem

Let  $\mathcal{A}$  be a square block matrix of order  $n$  over the ring  $R$ ; that is,

$$\mathcal{A} = \begin{pmatrix} A & C \\ B & D \end{pmatrix},$$

where  $A$  is a square block of order  $s$ , ( $1 < s < n$ ), the determinant of which,  $\delta_s$ , is neither zero nor a zero divider in  $R$ . Then, the adjoint matrix  $\mathcal{A}^*$  can be written as the product

$$\mathcal{A}^* = \begin{pmatrix} \delta_s^{-1}\delta_n I & -\delta_s^{-1}FC \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & G \end{pmatrix} \begin{pmatrix} I & 0 \\ -B & \delta_s I \end{pmatrix} \begin{pmatrix} F & 0 \\ 0 & I \end{pmatrix}, \quad (*)$$

where  $F = A^*$ ,  $G = \delta_s^{-n+s+1} \mathcal{A}_n^{(s+1)*}$ ,  $I$  is the identity matrix and we have the identity

$$\mathcal{A}_n^{(s+1)} = \delta_s D - BFC.$$

## Theorem

Let  $\mathcal{A}_n^{(s+1)}$  be a square block matrix of order  $n - s$ , ( $s > 0$ ,  $n - s > 2$ ), over the ring  $R$ ; that is,

$$\mathcal{A}_n^{(s+1)} = \begin{pmatrix} \mathbf{A} & \mathbf{C} \\ \mathbf{B} & \mathbf{D} \end{pmatrix},$$

where  $\mathbf{A}$  is a square block of order  $t - s$ , ( $1 < s < t < n$ ), and  $\delta_s$  and  $\delta_t$  are neither zero nor zero divisors in  $R$ . Then, the matrix  $\delta_s^{-n+s+1} \mathcal{A}_n^{(s+1)*}$  can be written as the product

$$\begin{pmatrix} \delta_t^{-1} \delta_n I & -\delta_t^{-1} \mathbf{FC} \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & \delta_s^{-1} \mathbf{G} \end{pmatrix} \begin{pmatrix} I & 0 \\ -\mathbf{B} & \delta_t I \end{pmatrix} \begin{pmatrix} \mathbf{F} & 0 \\ 0 & I \end{pmatrix}, (**)$$

where  $\mathbf{F} = \delta_s^{-t+s+1} \mathcal{A}_t^{(s+1)*}$ ,  $\mathbf{G} = \delta_t^{-n+t+1} \mathcal{A}_n^{(t+1)*}$ ,  $I$  is the identity matrix and we have the identity

$$\mathcal{A}_n^{(t+1)} = \delta_s^{-1} (\delta_t \mathbf{D} - \mathbf{BFC}).$$

**Remark 1:** If  $n = s + 2$ , then,  $\mathcal{A}_n^{(s+1)*} = \begin{pmatrix} a_{n,n}^{s+1} & -a_{n-1,n}^{s+1} \\ -a_{n,n-1}^{s+1} & a_{n-1,n-1}^{s+1} \end{pmatrix}$ .

And if  $n = s + 1$ , then  $\mathcal{A}_n^{(s+1)*} = 1$ .

**Remark 2:** In the factorization (\*)  $\delta_n$  can be found from Sylvester's identity  $\delta_n = \delta_s^{-n+s+1} \det \mathcal{A}_n^{(s+1)}$ . And if  $n = s + 2$ , then  $\delta_n = \det \mathbf{G}$ . In an analogous way we can find  $\delta_t$  and  $\delta_n$ , in the factorization (\*\*):

The dimensions of the upper left block  $A$  (of the initial square block matrix  $\mathcal{A}$ ) may be chosen arbitrarily. The case will be examined when the dimensions of block  $A$  are powers of two. Let  $n$  be the order of the matrix  $\mathcal{A}$ ,  $2^h < n \leq 2^{h+1}$  and assume that all minors  $\delta_{2^i}$ ,  $i = 1, 2, \dots$  are not zero or zero divisors of the ring  $R$ . According to Theorems 1 and 2 we are going to sequentially compute adjoint matrices for the upper left blocks of order  $2, 4, 8, 16, \dots$  of matrix  $\mathcal{A}$ .



1. For the block of order 2 we have:

$$A_{2,2}^2 = (a_{i,j})_{i,j=1,2}, \quad \delta_2 = \det A_{2,2}^1,$$

$$A_{2,2}^{2*} = \begin{pmatrix} a_{2,2} & -a_{1,2} \\ -a_{2,1} & a_{1,1} \end{pmatrix}.$$

2. For the block of order 4 we have:

$$A_{4,4}^{4*} = \begin{pmatrix} \delta_2^{-1} \delta_4 I & -\delta_2^{-1} FC \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & G \end{pmatrix} \begin{pmatrix} I & 0 \\ -B & \delta_2 I \end{pmatrix} \begin{pmatrix} F & 0 \\ 0 & I \end{pmatrix},$$

$$F = A_{2,2}^{2*}, \quad B = (a_{i,j})_{j=1,2}^{i=3,4},$$

$$C = (a_{i,j})_{j=3,3}^{i=1,2}, \quad D = (a_{i,j})_{i,j=3,4}, \quad \mathcal{A}_4^{(3)} = \delta_2 D - BFC = (a_{i,j}^3)_{i,j=3,4},$$

$$G = \delta_2^{-1} \mathcal{A}_4^{(3)*}, \quad \delta_4 = \det G.$$

3. For the block of order 8 we have:

$$A_{8,8}^{8*} = \begin{pmatrix} \delta_4^{-1}\delta_8 I & -\delta_4^{-1}FC \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & G \end{pmatrix} \begin{pmatrix} I & 0 \\ -B & \delta_4 I \end{pmatrix} \begin{pmatrix} F & 0 \\ 0 & I \end{pmatrix},$$

$$F = A_{4,4}^{4*}, \quad B = (a_{i,j})_{j=1,\dots,4}^{i=5,\dots,8},$$

$$C = (a_{i,j})_{j=5,\dots,8}^{i=1,\dots,4},$$

$$D = (a_{i,j})_{i,j=5,\dots,8},$$

$$G = \begin{pmatrix} \delta_6^{-1}\delta_8 I & -\delta_6^{-1}\mathbf{F}\mathbf{C} \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & \delta_4^{-1}\mathbf{G} \end{pmatrix} \begin{pmatrix} I & 0 \\ -\mathbf{B} & \delta_6 I \end{pmatrix} \begin{pmatrix} \mathbf{F} & 0 \\ 0 & I \end{pmatrix},$$

$$\mathcal{A}_8^{(5)} = \delta_4 D - BFC = (a_{i,j}^5)_{i,j=5,\dots,8},$$

$$\mathbf{F} = \delta_4^{-3} \mathcal{A}_6^{(5)*},$$

$$\delta_6 = \det \mathbf{F}, \quad \mathbf{B} = (a_{i,j}^5)_{j=5,6}^{i=7,8},$$

$$\mathbf{C} = (a_{i,j}^5)_{j=7,8}^{i=5,6},$$

$$\mathbf{D} = (a_{i,j}^5)_{i,j=7,8}, \quad \mathcal{A}_8^{(7)} = \delta_4^{-1}(\delta_6 \mathbf{D} - \mathbf{BFC}) = (a_{i,j}^7)_{i,j=7,8},$$

$$\mathbf{G} = \delta_6^{-1} \mathcal{A}_8^{(7)*},$$

$$\delta_8 = \det \mathbf{G}.$$

Example:

$$A = \begin{pmatrix} 0 & 2 & -2 & 2 \\ 1 & -3 & 1 & -2 \\ 3 & 0 & -3 & 0 \\ -1 & 3 & -1 & 1 \end{pmatrix}.$$

1. For the second order block we have:

$$A_{0,2}^{0,2} = \begin{pmatrix} 0 & 2 \\ 1 & -3 \end{pmatrix}, \alpha^2 = \det A_{0,2}^{0,2} = -2, A_{0,2}^{0,2*} = \begin{pmatrix} -3 & -2 \\ -1 & 0 \end{pmatrix}.$$

2. For the fourth order block we have:

$$F = A_{0,2}^{0,2*}, B = \begin{pmatrix} 3 & 0 \\ -1 & 3 \end{pmatrix}, C = \begin{pmatrix} -2 & 2 \\ 1 & -2 \end{pmatrix}, D = \begin{pmatrix} -3 & 0 \\ -1 & 1 \end{pmatrix},$$

$$A_{2,4}^{3;2,4} = \alpha^2 D - BFC = \begin{pmatrix} -6 & 6 \\ 0 & 2 \end{pmatrix}, G = (\alpha^2)^{-1} A_{2,4}^{3;2,4*} = \begin{pmatrix} -1 & 3 \\ 0 & 3 \end{pmatrix},$$

$$\alpha^4 = (\alpha^2)^{-1} \det A_{2,4}^{3;2,4} = 6,$$

$$A^* = \begin{pmatrix} (\alpha^2)^{-1} \alpha^4 I & -(\alpha^2)^{-1} FC \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & G \end{pmatrix} \begin{pmatrix} I & 0 \\ -B & \alpha^2 I \end{pmatrix} \begin{pmatrix} F & 0 \\ 0 & I \end{pmatrix}$$

$$\begin{aligned}
&= \begin{pmatrix} -3 & 0 & 2 & -1 \\ 0 & -3 & 1 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 3 \\ 0 & 0 & 0 & 3 \end{pmatrix} \\
&\times \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -3 & 0 & -2 & 0 \\ 1 & -3 & 0 & -2 \end{pmatrix} \begin{pmatrix} -3 & -2 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} -9 & -12 & 4 & -6 \\ -6 & -6 & 2 & 0 \\ -9 & -12 & 2 & -6 \\ 0 & -6 & 0 & -6 \end{pmatrix}.
\end{aligned}$$

## Complexity Estimation

Let  $\gamma n^\beta + o(n^\beta)$  be an asymptotic estimation of the number of operations for multiplying two matrices of order  $n$ . Then the complexity of computing the adjoint matrix of order  $n = 2^p$  is

$$F(n) = 6\gamma n^\beta \frac{1 - (n/2)^{1-\beta}}{2^\beta - 2} + o(n^\beta)$$



## Часть III

# Characteristic polynomial

In the case of an arbitrary commutative ring, the best algorithms for computing the characteristic polynomial are Chistov's algorithm [6] and the improved Berkowitz algorithm [3]. The complexity of these methods is  $O(n^{\beta+1} \log n)$ . We present the best method to date — for computations in commutative domains — which has complexity  $O(n^3)$ .

Let  $A = (a_{ij})$  be an  $n \times n$  matrix over the ring  $R$ . If all the diagonal minors  $\delta_k$  ( $k = 1, \dots, n - 1$ ) of matrix  $A$  are *not* zero, then the following identity holds

$$A_u = \tilde{L}A,$$

where  $A_u$  is an upper triangular matrix and  $\tilde{L}$  is a lower triangular matrix with determinant different from zero, such that

$$\tilde{L} = \mathbf{D}_{n-2}^{-1} \tilde{\mathbf{L}}_{n-1} \cdots \mathbf{D}_1^{-1} \tilde{\mathbf{L}}_2 \tilde{\mathbf{L}}_1$$

$\tilde{\mathbf{L}}_k = \text{diag}(I_{k-1}, \tilde{L}_k)$ ,  $\mathbf{D}_k = \text{diag}(I_k, D_k)$ , where  $I_k$  is the identity matrix of order  $k$ ,  $D_k = \delta_k I_{n-k}$ ,

$$L_k = \begin{pmatrix} \delta_k & 0 \\ v_k & I_{n-k} \end{pmatrix}, \tilde{L}_k = \begin{pmatrix} 1 & 0 \\ -v_k & \delta_k I_{n-k} \end{pmatrix}$$

$v_k = (a_{k+1,k}^k, \dots, a_{n,k}^k)^T$ ,  $A_u = (a_{i,j}^{(n)})$  is an  $n \times n$  matrix, and  
 $a_{i,j}^{(n)} = a_{i,j}^i$ , for  $i \leq j$ ,  $a_{i,j}^{(n)} = 0$ , for  $i > j$ .  
 The proof is based on Sylvester's identity

$$a_{k-1,k-1}^{k-1} a_{i,j}^{k+1} = a_{k,k}^k a_{i,j}^k - a_{i,k}^k a_{k,j}^k.$$

The factorization of matrix  $A$  into upper and lower triangular matrices is the result of the forward direction part — of the forward and backward direction algorithm.

Let  $A_u^{(k)} = (a_{i,j}^{(k)})$  be an  $n \times n$  matrix,  $k = 1, \dots, n$ , with  $a_{i,j}^{(k)} = a_{i,j}^i$  for  $i \leq j < k$ ,  $a_{i,j}^{(k)} = a_{i,j}^k$ ,  $i \geq k, j \geq k$ , and the remaining elements zero. Then  $A_u = \tilde{L}A$  reduces to the identities

$$A_u^{(2)} = \tilde{\mathbf{L}}_1 A; \quad A_u^{(k+1)} = \mathbf{D}_{k-1}^{-1} \tilde{\mathbf{L}}_k A_u^{(k)}, \quad k = 2, \dots, n-1,$$

which subsequently enable the computation of matrices  $A_u^{(k)}$ ,  $k = 2, 3, \dots, n$ , such that all the elements of the matrices  $\mathbf{D}_k$  and  $\tilde{\mathbf{L}}_k$  are elements of the matrix  $A_u^{(k)}$ .

The requirement that the diagonal minors  $\delta_k$  ( $k = 1, 2, \dots, n - 1$ ) be different from zero may be weakened. If a diagonal minor  $\delta_k$  of order  $k$  is equal to zero, and in column  $v_k$  there is a nonzero element  $a_{i,k}^k$ , then rows  $i$  and  $k$  must be interchanged; that is, multiply on the left the matrix of interchanges  $P_k = P_{(i,k)} = I_n + E_{ik} + E_{ki} - E_{kk} - E_{ii}$ , where  $E_{ik}$  denotes a matrix in which all elements are zero except element  $(i, k)$ , which is equal to one.

And if  $\delta_k = 0$  and  $v_k = 0$ , then necessarily  $P_k = \tilde{\mathbf{L}}_k = \mathbf{D}_{k-1} = I_n$ ,  
 $\mathbf{D}_k = \mathbf{D}_{k-1}$ .

The factorization formula remains as before, only now

$$\tilde{\mathbf{L}}_k = \text{diag}(I_{k-1}, \tilde{L}_k)P_k.$$

Note the following identities, which will be subsequently needed:

$$\tilde{L}_k L_k = D_k, \quad \tilde{L} L = T,$$

where

$$L = \mathbf{L}_1 \mathbf{L}_2 \cdots \mathbf{L}_{n-1},$$

$\mathbf{L}_k = P_k^{-1} \text{diag}(I_{k-1}, L_k)$ ,  $T$  is a diagonal matrix defined by,  $T = S_1 S_2$ , where  $S_1 = \text{diag}(1, S)$ , and  $S_2 = \text{diag}(S, 1)$ , with  $S = \text{diag}(\delta_1, \delta_2, \dots, \delta_{n-1})$ .

To indicate the matrix  $A$  from which a given triangular or diagonal matrix was computed, we write  $L = L(A)$ ,  $T = T(A)$ .



Let  $A = \begin{pmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{c} & \mathbf{d} \end{pmatrix}$  be a matrix over  $R$  with blocks  $\mathbf{a}$  of order  $p \times p$  and  $\mathbf{d}$  of order  $n \times n$ . We will call matrix  $A$  upper  $p$ -triangular, if the block  $(\mathbf{c}, \mathbf{d})$  looks like an upper triangular matrix. We will denote with calligraphic letters block-diagonal matrices of order  $(n + p) \times (n + p)$  of the type  $\text{diag}(I_p, G) = \mathcal{G}$ , where  $G$  is a  $p \times p$  matrix.

Let  $G$  be some  $p \times p$  matrix and let  $\tilde{L} = \tilde{L}((\mathbf{c}, \mathbf{d})\mathcal{G})$ , and  $T = T(((\mathbf{c}, \mathbf{d})\mathcal{G}))$ . If we take now  $G = L$ ,  $\tilde{\mathcal{L}} = \text{diag}(I_p, \tilde{L})$ ,  $\mathcal{L} = \text{diag}(I_p, L)$ , then the matrix

$$A_u = \tilde{\mathcal{L}}A\mathcal{L}$$

will become an upper  $p$ -triangular matrix, and matrix  $T^{-1}A_u$  will be similar to  $A$ .

The cofactors  $L$  and  $\tilde{L}$  of the matrix can be computed sequentially. Since  $((\mathbf{c}, \mathbf{d})\mathcal{G}) = (\mathbf{c}, \mathbf{d}L)$  and the first  $p$  of the columns of the matrix  $(\mathbf{c}, \mathbf{d}L)$  constitute block  $\mathbf{c}$  and are independent from  $L$ , then using them we can compute sequentially the first  $p$  cofactors of the matrix  $\tilde{L} : \tilde{\mathbf{L}}_1, \mathbf{D}_1, \tilde{\mathbf{L}}_2, \dots, \mathbf{D}_{p-1}, \tilde{\mathbf{L}}_p$ . From these we can write the first  $p$  cofactors of matrix  $L$ , can compute  $p$  columns of the matrix  $\mathbf{d}L$  and after that the following  $p$  cofactors of matrix  $\tilde{L}$ , etc. For  $p = 1$  we obtain a quasi-triangular matrix, that is a matrix with zero elements under the second diagonal, which is obtained by the elements  $a_{2,1}, a_{3,2}, \dots, a_{n,n-1}$ .

Let us denote by  $A_k$  ( $1 \leq k \leq n$ ) the corner minors of order  $k$  of the quasi-triangular matrix  $A = (a_{i,j})$ ,  $a_{i,j} = 0$  for  $i \geq 2, j \leq i - 1$ , and assume  $A_0 = 1$ . Then its determinant can be computed as shown

$$\det(A_n) = a_{nn} \det(A_{n-1}) + \sum_{i=1}^{n-1} a_{i,n} \det(A_{i-1}) \prod_{j=i+1}^{n-1} (-a_{j,j-1}).$$

The complexity of this method is  $\frac{5}{3}n^3 + O(n^2)$  — multiplicative operations.

**Example.**

Let  $A = \begin{pmatrix} 0 & 2 & 0 & 1 & 1 \\ 2 & 1 & 1 & 0 & 2 \\ 0 & 3 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 2 & 1 & 0 & 0 & 2 \end{pmatrix}$ , then we obtain:

$$\tilde{\mathcal{L}}_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & -1 & 0 & 2 & 0 \\ 0 & -2 & 0 & 0 & 2 \end{pmatrix}, \quad \mathcal{L}_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 & 1 \end{pmatrix},$$

$$A^{(2)} = \tilde{\mathcal{L}}_1 A \mathcal{L}_1 = \begin{pmatrix} 0 & 7 & 0 & 1 & 1 \\ 2 & 6 & 1 & 0 & 2 \\ 0 & 12 & 2 & 0 & 0 \\ 0 & -4 & -1 & 2 & -2 \\ 0 & 0 & -2 & 0 & 0 \end{pmatrix},$$

$$\tilde{\mathcal{L}}_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 4 & 12 & 0 \\ 0 & 0 & 0 & 0 & 12 \end{pmatrix}, \quad \mathcal{L}_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 12 & 0 & 0 \\ 0 & 0 & -4 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$A^{(3)} = \text{diag}^{-1}(1, 1, 1, 2, 2)\tilde{\mathcal{L}}_2 A^{(2)} \mathcal{L}_2 = \begin{pmatrix} 0 & 7 & -4 & 1 & 1 \\ 2 & 6 & 12 & 0 & 2 \\ 0 & 12 & 24 & 0 & 0 \\ 0 & 0 & -72 & 12 & -12 \\ 0 & 0 & -144 & 0 & 0 \end{pmatrix},$$

$$\tilde{\mathcal{L}}_3 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 144 & -72 \end{pmatrix}, \quad \mathcal{L}_3 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -72 & 0 \\ 0 & 0 & 0 & -144 & 1 \end{pmatrix},$$

$$\begin{aligned}
 A^{(4)} &= \text{diag}^{-1}(1, 1, 1, 1, 12) \tilde{\mathcal{L}}_2 A^{(2)} \mathcal{L}_2 \\
 &= \begin{pmatrix} 0 & 7 & -4 & -216 & 1 \\ 2 & 6 & 12 & -288 & 2 \\ 0 & 12 & 24 & 0 & 0 \\ 0 & 0 & -72 & 864 & -12 \\ 0 & 0 & 0 & 10368 & -144 \end{pmatrix}.
 \end{aligned}$$

The 1-triangular matrix  $A^{(4)}$  is similar to  $\text{diag}(1, 2, 24, -864, -72)A$ . The characteristic polynomial of the matrix  $A$  and the matrix  $\text{diag}(1, \frac{1}{2}, \frac{1}{24}, \frac{-1}{864}, \frac{-1}{72})A^{(4)}$  is equal to  $-x^5 + x^4 + 3x^3 - 22x^2 + 6x + 12$ .

Часть IV

Conclusion



For computations over commutative domains we have the following results:

- The complexity of the  $O(n^3)$  methods (FB) and (OP) for solving systems of linear equations of size  $n \times m$  is

$$M_{(FB)} = (1/2)(4n^2m - 2n^3 - 2nm - 3n^2 - 2m + 7n - 2),$$

$$M_{(OP)} = (1/6)(12n^2m - 8n^3 - 6nm - 9n^2 - 12m + 8n + 12).$$

Suppose that the complexity of the given method for matrix multiplications is  $\gamma n^\beta + o(n^\beta)$ , where  $\gamma$  and  $\beta$  are constants, and  $n$  is the order of the matrix. Then, the complexity of the recursive methods for solving systems of size  $n \times m$  is

$$S(n, m) = \gamma \frac{n^\beta}{2^\beta} \left[ \left(4 \frac{m}{n} - 2\right) \frac{1 - n^{2-\beta}}{1 - 2^{2-\beta}} - \frac{1 - n^{1-\beta}}{1 - 2^{1-\beta}} \right] + o(n^{\beta-1}m).$$







- The complexity of the method for the computation of the determinant of a matrix of order  $n$  is  $S(n, n)$ . The complexity of the method for the computation of the kernel of a linear operator is  $S(n, m)$ .
- The complexity of the method for the computation and the factorization of the adjoint matrix is

$$F(n) = 6\gamma n^\beta \frac{1 - (n/2)^{1-\beta}}{2^\beta - 2} + o(n^\beta)$$

- Finally, the complexity of the best method we know today for the computation of the characteristic polynomial of a matrix of order  $n$  is  $\frac{5}{3}n^3 + O(n^2)$ .

## Часть V

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