Symbolic-Algebraic Methods for Linear Partial Differential Operators (LPDOs)

Franz Winkler

joint with Ekaterina Shemyakova

RISC

Research Institute for Symbolic Computation

Johannes Kepler Universität · Linz · Austria

Outline

1. Definitions
2. Elimination theory for LPDOs
3. Laplace Transformation Method
   - Generalizations and Variations of the Laplace Method
4. First Constructive Factorization Algorithm: Grigoriev-Schwarz
5. Non-uniqueness Problem of Factorizations
   - A New Definition of a Factorization: Li,Schwarz,Tsarev, etc.
6. Obstacles to Factorizations
7. Factorization with possibly NON-coprime Symbols of Factors
8. Bivariate Parametric Factorizations of Orders $2, 3, 4$
9. Invariants
10. Generating Set of Invariants for Bivariate Hyperbolic LPDOs of Third-Order
11. A Maple-Package for LPDOs with Parametric Coefficients

Outline

1. Definitions
2. Elimination theory for LPDOs
3. Laplace Transformation Method
   - Generalizations and Variations of the Laplace Method
4. First Constructive Factorization Algorithm: Grigoriev-Schwarz
5. Non-uniqueness Problem of Factorizations
   - A New Definition of a Factorization: Li,Schwarz,Tsarev, etc.
6. Obstacles to Factorizations
7. Factorization with possibly NON-coprime Symbols of Factors
8. Bivariate Parametric Factorizations of Orders $2, 3, 4$
9. Invariants
10. Generating Set of Invariants for Bivariate Hyperbolic LPDOs of Third-Order
11. A Maple-Package for LPDOs with Parametric Coefficients

Definitions

Differential field:

Consider a field K with a set

$$\Delta = \{\partial_1, \dots, \partial_n\}$$

of derivations acting on it. For all elements of K these derivations satisfy:

- $\partial_i(a + b) = \partial_i(a) + \partial_i(b)$,
- $\partial_i(ab) = \partial_i(a)b + a\partial_i(b)$,
- $\partial_i(\partial_j(a)) = \partial_j(\partial_i(a))$.

Ring of linear differential operators:

$$K[D] = K[D_1, \dots, D_n] \ ,$$

where $D_1, \dots, D_n$ correspond to $\partial_1, \dots, \partial_n$, respectively, and

- $D_i \circ D_j = D_i \circ D_j$,
- $D_i \circ f = f \circ D_i + \partial_i(f) \quad \forall f \in K$.

Symbol of an LPDO:

Any operator $L \in K[D]$ is of the form

$$L = \sum_{|J| \leq d} a_J D^J \ , \tag{1}$$

where $a_J \in K, \ J \in \mathbf{N}^n$. Then the symbol is the polynomial

$$\mathrm{Sym}_L = \sum_{|J|=d} a_J X^J \ .$$

Hyperbolic LPDO:

An operator is *hyperbolic* if its symbol is completely factorable (all factors are of first order) and each factor has multiplicity one.

i-th component of L

is the sum of all components in (1) of order i.

The operation of transposition

is

$$L \to L^t(f) = \sum_{|J| \le d} (-1)^{|J|} D^J(a_J f) \ ,$$

where $f \in K$.

The gauge transformations:

$$L \to g^{-1} \circ L \circ g \quad , \ g \in K^* \ ,$$

where K^* denotes the set of invertible elements in K.

Invariant:

an algebraic differential expression of coefficients of LPDOs of certain class which is unaltered by these transformations.
Trivial examples of an invariant w.r.t. the gauge transformations are coefficients of the symbol of the operator.

Outline

1. Definitions
2. Elimination theory for LPDOs
3. Laplace Transformation Method
   - Generalizations and Variations of the Laplace Method
4. First Constructive Factorization Algorithm: Grigoriev-Schwarz
5. Non-uniqueness Problem of Factorizations
   - A New Definition of a Factorization: Li,Schwarz,Tsarev, etc.
6. Obstacles to Factorizations
7. Factorization with possibly NON-coprime Symbols of Factors
8. Bivariate Parametric Factorizations of Orders $2, 3, 4$
9. Invariants
10. Generating Set of Invariants for Bivariate Hyperbolic LPDOs of Third-Order
11. A Maple-Package for LPDOs with Parametric Coefficients

Definition

Let

- K be a field,
- $\Delta = \{\delta_1, \cdots, \delta_m\}$ be a set of derivations on K,
- $\Sigma = \{\sigma_1, \cdots, \sigma_n\}$ be a set of automorphisms of K, s.t.

$$\alpha(\beta(x)) = \beta(\alpha(x)) \ ,$$

$\forall \alpha, \beta \in \Delta \cup \Sigma$ and $x \in K$.

Then K is called a *difference-differential field* with the basic set of derivations Δ and the basic set of automorphisms Σ, or shortly a *Δ-Σ-field*.

Definition

Let

- K be a Δ-Σ-field,
- Λ be the free commutative semigroup of words over Δ and $\tilde{\Sigma}$ (Σ with inverses).

Then an expression of the form

$$\sum_{\lambda \in \Lambda} a_\lambda \lambda \ ,$$

where $a_\lambda \in K$ and only finitely many coefficients a_λ are non-zero, is called a *difference-differential operator* (or shortly a *Δ-Σ-operator*) over K.

The ring of all Δ-Σ-operators over a Δ-Σ-field K is called the *ring of difference-differential operators* (or shortly the *ring of Δ-Σ-operators*) over K.

Insa, Pauer: a Gröbner basis theory for differential operators in *"Gröbner Bases and Applications"*, Buchberger, Winkler (eds.), Cambridge Univ. Press (1998)

Levin (JSC 2000 and 2007) and Zhou,Winkler (Proc. ISSAC 2006): an extension to difference-differential operators.

Outline

1. Definitions
2. Elimination theory for LPDOs
3. Laplace Transformation Method
   - Generalizations and Variations of the Laplace Method
4. First Constructive Factorization Algorithm: Grigoriev-Schwarz
5. Non-uniqueness Problem of Factorizations
   - A New Definition of a Factorization: Li,Schwarz,Tsarev, etc.
6. Obstacles to Factorizations
7. Factorization with possibly NON-coprime Symbols of Factors
8. Bivariate Parametric Factorizations of Orders $2, 3, 4$
9. Invariants
10. Generating Set of Invariants for Bivariate Hyperbolic LPDOs of Third-Order
11. A Maple-Package for LPDOs with Parametric Coefficients

Laplace Transformation Method

Laplace (1749 - 1827)

Darboux (1842 - 1917)

is the oldest (the end of the 18th century) algebraic method of integration of PDEs [Laplace, Euler, Darboux, Forsyth, Goursat].

The method works for 2nd-order linear hyperbolic equations on the plane, which have the normalized form

$$z_{xy} + az_x + bz_y + c = 0,$$

where $a = a(x, y), b = b(x, y), c = c(x, y)$.

The corresponding differential operator

$$L = D_x \circ D_y + aD_x + bD_y + c$$

can be rewritten as

$$L = (D_x + b) \circ (D_y + a) + h = (D_y + a) \circ (D_x + b) + k,$$

where $h = c - a_x - ab$, $k = c - b_y - ab$.

h and k are known as *the Laplace invariants*.

Note that

L is factorable if and only if $h = 0$ or $k = 0$.

If L is factorable, then the equation $L(z) = 0$ is integrable.

Indeed, if L is factorable, solution of $L(z) = 0$ is reduced to the problem of the integration of the two first order equations:

$$\begin{cases} (D_x + b)(z_1) = 0, \\ (D_y + a)(z) = z_1. \end{cases}$$

Accordingly one gets the general solution of $z_{xy} + az_x + bz_y + c = 0$:

$$z = \left(A(x) + \int B(y) e^{\int ady - bdx} dy\right) e^{-\int ady}$$

with two arbitrary functions $A(x)$ and $B(y)$.

What do we do if L is not factorable?

Step 1. Apply Laplace transformations

$L \to L_1$ and $L \to L_{-1}$, which are defined by the substitutions

$$z_1 = (D_y + a)(z), \quad z_{-1} = (D_x + b)(z) \ .$$

and can be applied if $h \neq 0$ and $k \neq 0$.

Lemma

Family of operators

$$\{D_{xy} + aD_x + bD_y + c \quad | \quad a = a(a,y) \ , \ b = b(x,y) \ , \ c = c(x,y)\}$$

admits (is closed w.r.t.) Laplace transformations.

$$L_1 = D_{xy} + \left(a - ln|h|_y\right) D_x + bD_y + c + b_y - a_x - bln|h|_y \ ,$$

$$L_{-1} = D_{xy} + aD_x + \left(b - ln|k|_x\right) D_y + c - b_y + a_x - aln|k|_x \ .$$

Step 2. Check whether the Laplace invariants of L_1 and L_{-1} are zero.

$$h_1 = 2h - k - \partial_{xy}(ln|h|), \qquad k_1 = h \neq 0,$$
$$h_{-1} = k \neq 0, \qquad k_{-1} = 2k - h - \partial_{xy}(ln|k|).$$

1. If $h_1 = 0$, then L_1 is factorable, and $L_1(z_1) = 0$ can be solved in quadratures. Then, using the inverse substitution

$$z = \frac{1}{h}(z_1)_{-1},$$

   we obtain the complete solution of the original equation $L(z) = 0$.
2. If $k_{-1} = 0$, then do analogously.
3. If $h_1 \neq 0$ and $k_{-1} \neq 0$, apply the Laplace transformations again.

General picture in generic case

2 chains:

$$\begin{array}{llll} \cdots \leftarrow L_{-2} \leftarrow L_{-1} \leftarrow & L, & & \\ & L & \rightarrow L_1 \rightarrow L_2 \rightarrow \dots \end{array}$$

Lemma

- $L = h^{-1}(L_1)_{-1}h$,
- the Laplace invariants do not change under such substitution.

Consequently, we have 1 chain:

$$\cdots \leftrightarrow L_{-2} \leftrightarrow L_{-1} \leftrightarrow L \leftrightarrow L_1 \leftrightarrow L_2 \leftrightarrow \dots,$$

and the corresponding chain of invariants

$$\cdots \leftrightarrow k_{-2} \leftrightarrow k_{-1} \leftrightarrow k \leftrightarrow h \leftrightarrow h_1 \leftrightarrow h_2 \leftrightarrow \dots$$

Recapitulation: the algorithm

1. iterates the Laplace transformations until one of elements of the chain of invariants vanishes,
2. in this case, one can solve the corresponding transformed equation in quadratures and transform the solution to the solution of the original equation.

Theorem [for ex. Goursat/Darboux]

If the chain of invariants if finite in both directions, then one may obtain a quadrature free expression of the general solution of the original equation.

Example

Consider Tsarev's equation

$$z_{xy} - \frac{n(n+1)}{(x+y)^2} z = 0 \ ,$$

for which the chain of operators has the length n in either direction.

$n = 1$, i.e. $L = D_{xy} - \frac{2}{(x+y)^2}$

the chain is

$$D_{xy} + \frac{2}{x+y} D_y - \frac{2}{(x+y)^2} \quad \leftrightarrow L \leftrightarrow \quad D_{xy} + \frac{2}{x+y} D_x - \frac{2}{(x+y)^2} \ ,$$

The corresponding chain of the Laplace invariants is

$$0 \leftrightarrow k \leftrightarrow h \leftrightarrow 0 \ .$$

$h_1 = 0$

Therefore, L_1 is factorable, and the equation $L_1(z_1) = 0$ can be analytically solved:

$$z_1 = \frac{1}{(x+y)^2}\Big(\int B(y)(x+y)^2 dy + A(x)\Big) .$$

Using the inverse substitution, compute the solution of the initial equation:

$$z = \frac{1}{h} D_x(z_1) =$$

$$\frac{1}{2}A(x) + \frac{1}{(x+y)}\Big((x+y)\int (x+y)B(y)dy - \int (x+y)^2 B(y)dy - A(x)\Big) .$$

Generalizations and Variations of the Laplace Method

Darboux:

An explicit integration method of <u>non-linear</u> 2nd-order <u>scalar</u> equations of the form

$$F(x, y, z, z_x, z_y, z_{xx}, z_{xy}, z_{yy}) = 0.$$

The idea is to consider a linearization of the equation. Then one applies the Laplace method.

Sokolov, Ziber, Startsev

proved that a second order hyperbolic non-linear equation is Darboux integrable if and only if both Laplace sequences of the linearized operator are finite.

Anderson, Juras, Kamran

generalized this to the case of the equations of the general form $F(x, y, z, z_x, z_y, z_{xx}, z_{xy}, z_{yy}) = 0$ as a consequence of their analysis of higher degree conservation laws for different types of partial differential equations.

Dini

suggested a generalization of the Laplace transformations for a certain class of 2-order operators in the space of arbitrary dimension. But no general statement was proved.

Tsarev

proved that for a generic second-order linear partial differential operator in 3-dimensional space,

$$L = \sum_{i+j+k \leq 2} a_{ijk}(x, y, z) D_x D_y D_z$$

there exist two Dini transformations $L \to L_1$ and $L \to L_{-1}$ under the assumption that its principal symbol factors.

Athorne and Yilmaz

proposed a special transformation for systems whose order coincides with the number of independent variables.

Roux

undertook a serious effort to generalize the classical theory to arbitrary order operators in two independent variables.

Tsarev

described another procedure, which generalizes the Laplace method to the case of arbitrary order hyperbolic operators.

Outline

1. Definitions
2. Elimination theory for LPDOs
3. Laplace Transformation Method
   - Generalizations and Variations of the Laplace Method
4. First Constructive Factorization Algorithm: Grigoriev-Schwarz
5. Non-uniqueness Problem of Factorizations
   - A New Definition of a Factorization: Li,Schwarz,Tsarev, etc.
6. Obstacles to Factorizations
7. Factorization with possibly NON-coprime Symbols of Factors
8. Bivariate Parametric Factorizations of Orders $2, 3, 4$
9. Invariants
10. Generating Set of Invariants for Bivariate Hyperbolic LPDOs of Third-Order
11. A Maple-Package for LPDOs with Parametric Coefficients

First Constructive Factorization Algorithm: Grigoriev-Schwarz

The algorithm extends the factorization of the symbol of an operator to a factorization of the operator, or concludes that there is no such factorization. It is close in nature to the Hensel lifting algorithm of factorizations of polynomials.

Theorem (Constructive Proof)

Let $L \in K[D]$, and its symbol be factored into two coprime factors:

$$Sym_L = S_1 \cdot S_2 \ .$$

Then there exists at most one factorization of the form

$$L = \left(\widehat{S}_1 + G\right) \circ \left(\widehat{S}_2 + H\right) \ ,$$

where $G, H \in K[D]$ and $\mathrm{ord}(G) < \mathrm{ord}(\widehat{S}_1)$, $\mathrm{ord}(H) < \mathrm{ord}(\widehat{S}_2)$.

Substitute

$$L = \sum_{i=0}^{d} L_i \ , \quad G = \sum_{i=0}^{k_1-1} G_i \ , \quad H = \sum_{i=0}^{k_2-1} H_i$$

where $d = \mathrm{ord}(L)$, $k_1 = \mathrm{ord}(\widehat{S}_1)$, $k_2 = \mathrm{ord}(\widehat{S}_2)$ into $L = \left(\widehat{S}_1 + G\right) \circ \left(\widehat{S}_2 + H\right)$:

$$\sum_{i=0}^{d} L_i = \left(\widehat{S}_1 + G_{k_1-1} + \cdots + G_0\right) \circ \left(\widehat{S}_2 + H_{k_2-1} + \cdots + H_0\right).$$

Equating the components of the orders $d-1$, $d-2$, $\ldots 0$, one gets a system of homogeneous polynomial equations:

$$
\begin{cases}
L_{d-1} = S_1 \cdot H_{k_2-1} + G_{k_1-1} \cdot S_2 \ , \\
L_{d-2} = S_1 \cdot H_{k_2-2} + G_{k_1-2} \cdot S_2 + P_{d-2} \ , \\
\dots \\
L_i = S_1 \cdot H_{k_2-(d-i)} + G_{k_1-(d-i)} \cdot S_2 + P_i \ , \quad (*) \\
\dots
\end{cases}
$$

where P_i are some expressions of H_{k_2-j}, G_{k_1-j} with $j < i$.

If we solve the system in descending order, the polynomials P_i can be considered as known.

Every equation $(*)$ is equivalent to a linear algebraic system in coefficients of the polynomials H_{i-k_1} and G_{i-k_2}. Since S_1 and S_2 are coprime, then there is at most 1 solution.
Thus, at every step one either gets the next components of H and G, or concludes that there is no factorization of the operator L that extends $\mathrm{Sym}_L = S_1 \cdot S_2$.

By induction on the number of factors one proves the following theorem:

Theorem

Let $L \in K[D]$, and

$$\mathrm{Sym}_L = S_1 \cdot S_2 \dots S_k \ ,$$

where $S_1, \dots, S_k$ are coprime. Then there exists at most one factorization

$$L = F_1 \circ \dots \circ F_k \ ,$$

such that

$$\mathrm{Sym}_{F_i} = S_i \ , \ i = 1, \dots k \ .$$

Example

Consider <u>non</u>-hyperbolic

$$L = D_{xyy} + D_{xx} + D_{xy} + D_{yy} + xD_x + D_y + x \ ,$$

which nonetheless has the factorization of the symbol

$$Sym_L = (X) \cdot (Y^2)$$

into coprime factors. The corresponding factorizations of L has the form

$$L = (D_x + G_0) \circ (D_{yy} + H_1 + H_0) \ ,$$

where $G_0 = r$, $H_1 = aD_x + bD_y$, and $H_0 = c$, where $r, a, b, c \in K$. Equate the components on the both sides of the equality:

$$
\begin{cases}
L_2 &= (aX + bY)X + rY^2 \ , \\
L_1 &= (c + ra + a_x)X + (b_x + rb)Y \ , \\
L_0 &= rc + c_x \ ,
\end{cases}
\tag{2}
$$

where L_i stands for the homogeneous polynomial corresponding to the component L_i of L, that is

$$
L_2 = X^2 + XY + Y^2, \quad L_1 = xX + Y, \quad L_0 = x \ .
$$

The first equation gives us $a = b = r = 1$. We plug this to the second equation, and get $c = x - 1$, that makes the last (third) equation of the system identity. Therefore operator L can be factored as follows:

$$
L = (D_x + 1) \circ (D_{yy} + D_x + D_y + x - 1) \ .
$$

Concluding Remark on Grigoriev-Schwarz's algorithm

It reveals a large class of LPDOs which factor uniquely.

In general, even for a given factorization of the symbol, factorization of LPDOs is not unique!

Outline

1 Definitions

2 Elimination theory for LPDOs

3 Laplace Transformation Method
- Generalizations and Variations of the Laplace Method

4 First Constructive Factorization Algorithm: Grigoriev-Schwarz

5 Non-uniqueness Problem of Factorizations
- A New Definition of a Factorization: Li,Schwarz,Tsarev, etc.

6 Obstacles to Factorizations

7 Factorization with possibly NON-coprime Symbols of Factors

8 Bivariate Parametric Factorizations of Orders $2, 3, 4$

9 Invariants

10 Generating Set of Invariants for Bivariate Hyperbolic LPDOs of Third-Order

11 A Maple-Package for LPDOs with Parametric Coefficients

Non-uniqueness Problem of Factorizations

Linear Ordinary Differential Operators

Loewy uniqueness theorem,
a fact. algorithm for LODOs over rational functions, etc.

Linear Partial Differential Operators

$$L = D_x^3 + xD_x^2 D_y + 2D_x^2 + (2x+2)D_x D_y + D_x + (2+x)D_y$$

[Landau] has two factorizations into different numbers of irreducible factors:

$$L = Q \circ Q \circ P = R \circ Q \ ,$$

for the operators

$$P = D_x + xD_y, \quad Q = D_x + 1, \quad R = D_{xx} + xD_{xy} + D_x + (2+x)D_y \ .$$

Note that the second-order operator R is absolutely irreducible, that is one cannot factor it into product of first-order operators with coefficients in any extension of $\mathbb{Q}(x, y)$.

A New Definition of a Factorization: Li,Schwarz,Tsarev, etc.

Denote by $< L >$ the left ideal generated by L.

Linear Ordinary Differential Operators:

principal ideal domain, thus,

$$\exists L \quad \text{s.t.} \quad < L_1 > \cap < L_2 > = < L > \; .$$

And so

$$lcm(L_1, L_2) = L \; .$$

Linear Partial Differential Operators:

the intersection ideal of two principal ideals is not necessarily principal.

Completely irreducible LPDO:

L is so, if

$$< L >=< L_1 > \cap \cdots \cap < L_k >$$

where L_i are irreducible. If this is the case, we say that

$$L = lcm(L_1, \ldots, L_k) \ .$$

Theorem

If $L_1, \ldots, L_k$ are right factors of L s.t.

$$\mathrm{Sym}(L) = \mathrm{Sym}(L_1) \ldots \mathrm{Sym}(L_k) \ ,$$

then

$$< L >=< L_1 > \cap \cdots \cap < L_k > \ .$$

Corollary

If L_i are irreducible, then L is completely irreducible.

Outline

1. Definitions
2. Elimination theory for LPDOs
3. Laplace Transformation Method
   - Generalizations and Variations of the Laplace Method
4. First Constructive Factorization Algorithm: Grigoriev-Schwarz
5. Non-uniqueness Problem of Factorizations
   - A New Definition of a Factorization: Li,Schwarz,Tsarev, etc.
6. Obstacles to Factorizations
7. Factorization with possibly NON-coprime Symbols of Factors
8. Bivariate Parametric Factorizations of Orders $2, 3, 4$
9. Invariants
10. Generating Set of Invariants for Bivariate Hyperbolic LPDOs of Third-Order
11. A Maple-Package for LPDOs with Parametric Coefficients

Obstacles to Factorizations

We study properties of factorizations for Grigoriev-Schwarz's class of LPDOs, i.e. symbols of factors are coprime.

Motivation: Laplace's incomplete factorizations for $\mathrm{ord} = 2$

$$L = D_x \circ D_y + aD_x + bD_y + c.$$

can be rewritten in the following ways:

$$L = (D_x + b) \circ (D_y + a) + h = (D_y + a) \circ (D_x + b) + k,$$

where

$$h = c - a_x - ab, \quad k = c - b_y - ab$$

are *the Laplace invariants*.

h and k, each of them is uniquely defined, and they together form a full system of invariants.

Definition (Generalization to $\mathrm{ord} = n$)

For $\mathrm{Sym}_L = S_1 \dots S_k$, we can always find a partial factorization:

$$L = L_1 \circ \cdots \circ L_k + R,$$

where $\mathrm{Sym}(L_i) = S_i$, $i = 1, \dots, k$, and R is of the smallest possible order. R is a common obstacle to factorizations of the type $(S_1)(S_2)\dots(S_k)$.

R is are neither unique, nor invariant.

Definition

The ring of obstacles to factorizations of the type $(S_1)\dots(S_k)$:

$$K(S_1,\dots,S_k)=K[X]/I\ ,\quad \text{where}\quad I=\left(\frac{\mathrm{Sym}_L}{S_1},\dots,\frac{\mathrm{Sym}_L}{S_k}\right)$$

is a homogeneous ideal, and $\mathrm{Sym}_L=S_1\dots S_k$.

Theorem

If all the S_i are pairwise coprime, then all common obstacles belong to the same class.

Definition

The class of common obstacles in $K(S_1,\dots,S_k)$ is the obstacle to factorization.

Theorems: Uniqueness, invariability, order $\leq(\mathrm{ord}(L)-2)$, generalized GS theorem, dimension of factorizable LPDOs, etc.

Outline

1. Definitions
2. Elimination theory for LPDOs
3. Laplace Transformation Method
   - Generalizations and Variations of the Laplace Method
4. First Constructive Factorization Algorithm: Grigoriev-Schwarz
5. Non-uniqueness Problem of Factorizations
   - A New Definition of a Factorization: Li,Schwarz,Tsarev, etc.
6. Obstacles to Factorizations
7. Factorization with possibly NON-coprime Symbols of Factors
8. Bivariate Parametric Factorizations of Orders $2, 3, 4$
9. Invariants
10. Generating Set of Invariants for Bivariate Hyperbolic LPDOs of Third-Order
11. A Maple-Package for LPDOs with Parametric Coefficients

Factorization, symbols of factors are NON-coprime

Parametric Factorizations

Linear Ordinary Differential Operators

Example (Classical)

$$D_{xx} = D_x \circ D_x = \left(D_x + \frac{1}{x + c}\right) \circ \left(D_x - \frac{1}{x + c}\right)$$

Linear Partial Differential Operators

Example (There are many, e.g. take operator from Landau' example)

$$\begin{aligned} L \;\; = \;\; & D_{xx}(D_x + xD_y) + 2D_{xx} + 2(x+1)D_{xy} + D_x + (x+1)D_y = \\ & \left(D_x + 1 + \frac{1}{x + c(y)}\right) \circ \left(D_x + 1 - \frac{1}{x + c(y)}\right) \circ (D_x + xD_y) \end{aligned}$$

Treatment of Parameters

Linear Ordinary Differential Operators

Possible number of parameters, description of the structure of families are known [Tsarev].

Linear Partial Differential Operators

Possible number of parameters, description of the structure of families for orders $2, 3, 4$ were found [Own Result].

Outline

1. Definitions
2. Elimination theory for LPDOs
3. Laplace Transformation Method
   - Generalizations and Variations of the Laplace Method
4. First Constructive Factorization Algorithm: Grigoriev-Schwarz
5. Non-uniqueness Problem of Factorizations
   - A New Definition of a Factorization: Li,Schwarz,Tsarev, etc.
6. Obstacles to Factorizations
7. Factorization with possibly NON-coprime Symbols of Factors
8. Bivariate Parametric Factorizations of Orders $2, 3, 4$
9. Invariants
10. Generating Set of Invariants for Bivariate Hyperbolic LPDOs of Third-Order
11. A Maple-Package for LPDOs with Parametric Coefficients

Bivariate Parametric Factorizations of Orders $2, 3, 4$

Definition

Reducible family: $F_1(T) \circ \cdots \circ F_k(T)$ if

- either the product of some number of first factors does not depend on parameter T,
- or if $F_1(T)$ ($F_k(T)$) has a left (right) factor that does not depend on T.

Otherwise, the family is irreducible.

Example (Used couple of slides above)

Reducible Family of Order 3 and Irreducible of Order 2:

$$L = \left(D_x + 1 + \frac{1}{x + c(y)}\right) \circ \left(D_x + 1 - \frac{1}{x + c(y)}\right) \circ \left(D_x + xD_y\right)$$

Reduce the amount of cases: irreducible families of factorizations

Definition

A factorization $L = F_1 \circ \cdots \circ F_k$ is of the factorization type $(S_1) \ldots (S_k)$, if $S_i = \mathrm{Sym}_{F_i}$ for all i.

Theorem

- *If there exists an irreducible family of the factorization type $(S_1)(S_2)(S_3)$ then there are also irreducible families of the factorization types $(S_1 S_2)(S_3)$ and $(S_1)(S_2 S_3)$.*
- *If there exists an irreducible family of the factorization type $(S_1)(S_2)$ then there is also an irreducible family of for the factorization type $(S_2)(S_1)$ (in fact, the latter family corresponds to the adjoint operator).*

Theorem

Operators of order two and three: a family exists only if a given operator is ordinary in some system of coordinates.
Operators of order four: a family exists only for factorizations of the types $(X)(Y)(XY)$, $(XY)(XY)$, $(X^2)(X^2)$ *(and symmetric to them).*

Structure of families

Theorem

Operators of order two: A family is unique for a given operator and depends on one parameter only.
Operators of order three: Any family depends by at most three (two) parameters if the number of factors in factorizations is three (two). Each of these parameters is a function of one variable.

The first non-trivial example of a parametric factorization of an LPDO of high order

Example

$$\begin{aligned} D_{xxyy} &= \left(D_x + \frac{\alpha}{y+\alpha x+\beta}\right)\left(D_y + \frac{1}{y+\alpha x+\beta}\right) \\ &\quad \left(D_{xy} - \frac{1}{y+\alpha x+\beta}(D_x+\alpha D_y)\right) , \end{aligned}$$

where $\alpha, \beta \in K \setminus \{0\}$.

Outline

1. Definitions
2. Elimination theory for LPDOs
3. Laplace Transformation Method
   - Generalizations and Variations of the Laplace Method
4. First Constructive Factorization Algorithm: Grigoriev-Schwarz
5. Non-uniqueness Problem of Factorizations
   - A New Definition of a Factorization: Li,Schwarz,Tsarev, etc.
6. Obstacles to Factorizations
7. Factorization with possibly NON-coprime Symbols of Factors
8. Bivariate Parametric Factorizations of Orders $2, 3, 4$
9. **Invariants**
10. Generating Set of Invariants for Bivariate Hyperbolic LPDOs of Third-Order
11. A Maple-Package for LPDOs with Parametric Coefficients

Invariants

Laplace invariants

For family of LPDOs of the form

$$L = D_{xy} + a(x,y)D_x + b(x,y)D_y + c(x,y)$$

the Laplace invariants h, k form a generating set of differential invariants with respect to the gauge transformations

$$L \mapsto g(x,y)^{-1} \circ L \circ g(x,y) \ .$$

Finding of a Generating Set of Invariants

is a classical problem of the classification of PDEs, also important for description of invariant properties.

Example

Equation of the form

$$z_{xy} + a(x,y)z_x + b(x,y)z_y + c(x,y)z = 0,$$

is equivalent to the wave equation

$$z_{xy} = 0$$

whenever $h = k = 0$.

Invariants for Hyperbolic Bivariate LPDOs

$\mathrm{ord} = 2$

There is a generating set consisting of two invariants: the Laplace invariants h and k.
For the more broader group of contact transformations there are also results [Anderson, Kamran, Moroz, etc.]

$\mathrm{ord} = 3$

- Symbol with Constant Coefficients: 4 invariants were determined, but they are not sufficient to form a generating set of invariants [Kartaschova].
- Arbitrary Symbol: an idea to get some invariants, but again insufficient to form a generating set of invariants [Tsarev].
- Arbitrary Symbol: 5 independent invariants are found which form a generating set of invariants [Shemyakova, Winkler].

Generating Set of Invariants for Hyperbolic Operator of Third-Order

If a bivariate third-order LPDO is hyperbolic it can be written in the following form after appropriate change of variables:

$$L = (pD_x + qD_y)D_xD_y + a_{20}D_x^2 + a_{11}D_{xy} + a_{02}D_y^2 + a_{10}D_x + a_{01}D_y + a_{00}\ , \quad (3)$$

where all coefficients are functions of x and y.

Fact number 1.

Operators of the form (3) admits Gauge Transformations $L \mapsto g(x_1, x_2)^{-1} \circ L \circ g(x_1, x_2)$, i.e. these transformations preserve operator's form (3).

Fact number 2.

As symbols are invariants under the Gauge Transformations, p and q are invariants for operators of the form (3).

Theorem

The following functions of coefficients are invariants, and together generate all possible differential invariants for the considered class of operators:

$$
\begin{aligned}
I_p &= p \ , \\
I_q &= q \ , \\
I_1 &= 2a_{20}q^2 - a_{11}pq + 2a_{02}p^2 \ , \\
I_2 &= \partial_x(a_{20})pq^2 - \partial_y(a_{02})p^2q + a_{02}p^2q_y - a_{20}q^2p_x \ , \\
I_3 &= a_{10}p^2 - a_{11}a_{20}p + a_{20}(2q_yp - 3qp_y) + a_{20}^2q - a_{11,y}p^2 + a_{11}p_yp + a_{20} \\
I_4 &= a_{01}q^2 - a_{11}a_{02}q + a_{02}(2qp_x - 3pq_x) + a_{02}^2p - a_{11,x}q^2 + a_{11}q_xq + a_{02} \\
I_5 &= a_{00}p^3q - p^3a_{02}a_{10} - p^2qa_{20}a_{01} + \\
&\quad (pI_1 - pq^2p_y + qp^2q_y)a_{20x} + (qq_xp^2 - q^2p_xp)a_{20y} \\
&\quad +(4q^2p_xp_y - 2qp_xq_yp + qq_{xy}p^2 - q^2p_{xy}p - 2qq_xpp_y)a_{20} \\
&\quad +(\frac{1}{2}p_{xy}p^2q - p_xp_ypq)a_{11} - \frac{1}{2}p^3qa_{11xy} + \frac{1}{2}a_{11x}p_yp^2q + \frac{1}{2}a_{11y}p_xp^2q \\
&\quad +p^2a_{02}a_{20}a_{11} + pqp_xa_{20}a_{11} - 2p_xq^2a_{20}^2 - 2p^2p_xa_{20}a_{02} \ .
\end{aligned}
$$

Thus, an operator

$$L' = D_x D_y(\widetilde{p} D_x + \widetilde{q} D_y) + \widetilde{a}_{20} D_x^2 + \widetilde{a}_{xx} D_x D_y + \widetilde{a}_{02} D_y^2 + \widetilde{a}_{10} D_x + \widetilde{a}_{01} D_y + \widetilde{a}_{00}$$

is equivalent to

$$L = D_x D_y(p D_x + q D_y) + a_{20} D_x^2 + a_{11} D_{xy} + a_{02} D_y^2 + a_{10} D_x + a_{01} D_y + a_{00} \ ,$$

if and only if the values of their invariants $p, q, I_1, I_2, I_3, I_4, I_5$ are equal correspondingly.

Outline

1. Definitions
2. Elimination theory for LPDOs
3. Laplace Transformation Method
   - Generalizations and Variations of the Laplace Method
4. First Constructive Factorization Algorithm: Grigoriev-Schwarz
5. Non-uniqueness Problem of Factorizations
   - A New Definition of a Factorization: Li,Schwarz,Tsarev, etc.
6. Obstacles to Factorizations
7. Factorization with possibly NON-coprime Symbols of Factors
8. Bivariate Parametric Factorizations of Orders $2, 3, 4$
9. Invariants
10. Generating Set of Invariants for Bivariate Hyperbolic LPDOs of Third-Order
11. A Maple-Package for LPDOs with Parametric Coefficients

A Maple-Package for LPDOs with Parametric Coefficients

Description

- The number of variables – Arbitrary.
- The orders of LPDOs – Arbitrary.
- Parameters – Arbitrary.
- Easy access to the coefficients of LPDOs.
- Application to a function $\to$ to a standard Maple PDE form.

Basic Procedures

- The basic arithmetic of LPDOs (addition, composition, mult. by a function on the left).
- Transposition and conjugation of LPDOs.
- Application to a function $\to$ to a standard Maple PDE form.
- Simplification Tools for coefficients.

More Advanced Possibilities

- Standard Laplace invariants.
- Standard Laplace Transformations.
- Standard Laplace Chain.
- Laplace Invariants for extended Schrödinger operators: $\Delta_2 + aD_x + bD_y + c$.
- Laplace Transformations for those.
- Laplace Chain for those.
- Full System of Invariants for operators $L_3 = D_x D_y (pD_x + qD_y) + \dots$
- Obstacles to factorizations of $2, 3$ orders $\rightarrow$ Grigoriev-Schwarz Factorization.