

**Analytic solutions of linear difference
equations, formal series, and bottom
summation**

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difference operators

Let E be the shift operator such that $E(f(k)) = f(k + 1)$ for sequences $f(k)$ where $k \in \mathbb{Z}$, and $E(\varphi(z)) = \varphi(z + 1)$ for analytic functions, $z \in \mathbb{C}$. Let

$$L = a_d(k)E^d + \cdots + a_1(k)E + a_0(k) \in \mathbb{C}(k)[E].$$

a summing operator

We say that an operator $R \in \mathbb{C}(k)[E]$ is a *summing operator* for L if

$$(E - 1) \circ R = 1 + M \circ L \quad (1)$$

for some $M \in \mathbb{C}(k)[E]$. We can assume w.l.g. that $\text{ord } R = \text{ord } L - 1 = d - 1$:

$$R = r_{d-1}(k)E^{d-1} + \cdots + r_1(k)E + r_0(k) \in \mathbb{C}(k)[E].$$

the discrete Newton-Leibniz formula

If a summing operator exists, then it can be constructed by the Accurate Summation algorithm [Abramov, van Hoeij, 1999] or, when $d = 1$, by Gosper's algorithm [Gosper, 1978]. At first glance, in those cases where $R \in \mathbb{C}(k)[E]$ exists, equality (1) gives an opportunity to use the discrete Newton-Leibniz formula (DNLF)

$$\sum_{k=v}^{w-1} f(k) = g(w) - g(v)$$

for all integers $v < w$, and for any sequence f such that $L(f) = 0$, taking $g = R(f)$.

Indeed, we can apply both sides of $(E - 1) \circ R = 1 + M \circ L$ to f .

This gives

$$(E - 1)(R(f)) = f + M(L(f)).$$

Set $g = R(f)$. Taking into account that $L(f) = \mathbf{0}$ we get

$$(E - 1)g = f,$$

or

$$g(k + 1) - g(k) = f(k),$$

and, as a consequence, the DNLF is applicable:

$$\begin{aligned} \sum_{k=v}^{w-1} f(k) &= g(v+1) - g(v) + g(v+2) - g(v+1) + \dots + g(w) - g(w-1) \\ &= g(w) - g(v) \end{aligned}$$

(the telescoping effect).

however...

However, it was shown that if R has rational-function coefficients which have poles in \mathbb{Z} , then this formula may give an incorrect result (an example will be demonstrated below). This gives rise to defects in many implementations of summation algorithms.

Example 1 Consider the sequence

$$f(k) = \frac{\binom{2k-3}{k}}{4^k}$$

which satisfies the first order equation

$$2(k+1)(k-2)f(k+1) - (2k-1)(k-1)f(k) = 0.$$

Though Gosper's algorithm succeeds on this sequence, producing $R(k) = \frac{2k\binom{k+1}{k-2}}{k-2}$, and $f(k)$ is defined for all $k \in \mathbb{Z}$, the discrete Newton-Leibniz formula

$$\sum_{k=0}^{w-1} f(k) = R(w)f(w) - R(0)f(0) = \frac{2w(w+1)\binom{2w-3}{w}}{(w-2)4^w}$$

is not correct: if we assume that the value of $\binom{2k-3}{k}$ is 1 when $k = 0$ and -1 when $k = 1$ (as is common practice in combinatorics) then the expression on the right gives the true value of the sum only at $w = 1$.

Suppose that L acts on analytic functions:

$$L = a_d(z)E^d + \cdots + a_1(z)E + a_0(z) \in \mathbb{C}(z)[E].$$

We consider the summing operator (if it exists) in the form

$$R = r_{d-1}(z)E^{d-1} + \cdots + r_1(z)E + r_0(z) \in \mathbb{C}(z)[E].$$

Let $\varphi(z)$ be a meromorphic solution of $L(y) = 0$.

But it turns out that if $\varphi(z)$ has no pole in \mathbb{Z} , then $R(\varphi)(z)$ has no pole in \mathbb{Z} as well, and we can use the DNLF to sum values $\varphi(k)$ for $k = v, v + 1, \dots, w$.

So such undesirable phenomena demonstrated in Example 1 cannot occur if the elements of the sequence under summation are the values $\varphi(k)$, $k \in \mathbb{Z}$, of an analytic function $\varphi(z)$, which satisfies (in the complex plane \mathbb{C}) the same difference equation with polynomial coefficients as does the original sequence (at integer points).

This follows from a stronger statement. The fact is that even if $\varphi(z)$ has some poles in \mathbb{Z} , the summation task can nevertheless be performed correctly.

the depth and the bottom of a merom. function

For any $k \in \mathbb{Z}$ the function $\varphi(z)$ can be represented by Laurent's series

$$\varphi(z) = c_{k, \rho_k} (z - k)^{\rho_k} + c_{k, \rho_k + 1} (z - k)^{\rho_k + 1} + \dots$$

with $\rho_k \in \mathbb{Z}$ and $c_{k, \rho_k} \neq 0$. If $L(\varphi) = 0$, then there exists the minimal element ρ in the set of all ρ_k , $k \in \mathbb{Z}$.

This ρ we call the *depth* of $\varphi(z)$ and denote it as $\text{depth}(\varphi)$.

We associate with $\varphi(z)$ the sequence $f(k)$ such that $f(k) = c_{k, \rho_k}$ if $\rho_k = \rho$, and $f(k) = 0$ otherwise. This $f(k)$ we call the *bottom* of $\varphi(z)$ and denote it as $\text{bott}(\varphi)$.

We illustrate the introduced notions by the following simple example.

It is well known that $\Gamma(z)$ has finite values when $z = 1, 2, \dots$, and has simple poles when $z = 0, -1, -2, \dots$

We have

$$\text{depth}(\Gamma) = -1,$$

and

$$\text{bott}(\Gamma)(k) = \begin{cases} 0, & \text{if } k > 0, \\ \frac{(-1)^{k+1}}{(-k-1)!}, & \text{if } k \leq 0. \end{cases}$$

If we consider $\Gamma(z)$ only in the half-plane $\text{Re}z > 0$, then its depth is 0, and the bottom is the sequence

$$f(k) = (k-1)!, \quad k = 1, 2, \dots$$

properties of the depth and the bottom

Proposition 1 *Let $L(\varphi) = 0$. Then*

$$L(\text{bott}(\varphi)) = 0.$$

Proposition 2 *Let $L(\varphi) = 0$, and R be a summing operator for L .
Then*

$$\text{depth}(\varphi) = \text{depth}(R(\varphi)).$$

the main result (the bottom summation)

Theorem 1 (On the bottom summation.) *Let $L(\varphi(z)) = 0$, and R be a summing operator for L . Let $\psi(z) = R(\varphi(z))$.*

Then the bottom summation formula

$$\sum_{k=v}^{w-1} \text{bott}(\varphi)(k) = \text{bott}(\psi)(w) - \text{bott}(\psi)(v)$$

is valid for any $v < w$. In particular, if φ has no pole in \mathbb{Z} (i.e., $\text{depth}(\varphi) \geq 0$), then the function $\psi(z)$ has no pole in \mathbb{Z} , and the discrete Newton-Leibniz formula

$$\sum_{k=v}^{w-1} \varphi(k) = \psi(w) - \psi(v)$$

is valid for any $v < w$.

Example 1 continued. Assume that the value of $\binom{2k-3}{k}$ is defined

as

$$\lim_{z \rightarrow k} \frac{\Gamma(2z-2)}{\Gamma(z+1)\Gamma(z-2)}; \quad (2)$$

this is a natural extension of the formula

$$\binom{2k-3}{k} = \frac{(2k-3)!}{k!(k-3)!}$$

for all $k \in \mathbb{Z}$.

Set

$$\varphi(z) = \frac{\Gamma(2z-2)}{\Gamma(z+1)\Gamma(z-2)4^z},$$

and

$$\psi(z) = \frac{2z(z+1)}{z-2} \varphi(z).$$

Limit (2) exists for all $k \in \mathbb{Z}$, and $\text{depth}(\varphi) = 0$.

Then the DNLF gives the correct result

$$\sum_{k=0}^{w-1} \frac{\Gamma(2k-2)}{\Gamma(k+1)\Gamma(k-2)4^k} = \frac{2w(w+1)\Gamma(2w-2)}{(w-2)\Gamma(w+1)\Gamma(w-2)4^w},$$

$w = 1, 2, \dots$

Earlier we assumed that the value of $\binom{2k-3}{k}$ is 1 when $k = 0$ and -1 when $k = 1$ (as is common practice in combinatorics). However

$$\lim_{z \rightarrow 0} \frac{\Gamma(2z - 2)}{\Gamma(z + 1)\Gamma(z - 2)} = \frac{1}{2} \neq 1$$

and

$$\lim_{z \rightarrow 1} \frac{\Gamma(2z - 2)}{\Gamma(z + 1)\Gamma(z - 2)} = -\frac{1}{2} \neq -1.$$

This example demonstrates a conflict between combinatorial and analytic definitions of the symbol $\binom{p}{q}$.

Example 2 The function $\varphi(z) = z\Gamma(z+1)$ satisfies the equation $L(y) = 0$ where $L = zE - (z+1)^2$. We have $R = \frac{1}{z}$, $\text{ord } R = 0$, and $\psi(z) = R(\varphi)(z) = \Gamma(z+1)$. Evidently $\varphi(z)$ has finite values when $z = 0, 1, \dots$, and has simple poles when $z = -1, -2, \dots$. We have $\text{depth}(\varphi) = \text{depth}(\psi) = -1$ and

$$\text{bott}(\varphi)(k) = \begin{cases} \frac{(-1)^{k+1}k}{(-k-1)!}, & \text{if } k < 0, \\ 0, & \text{if } k \geq 0, \end{cases}$$

$$\text{bott}(\psi)(k) = \begin{cases} \frac{(-1)^{k+1}}{(-k-1)!}, & \text{if } k < 0, \\ 0, & \text{if } k \geq 0. \end{cases}$$

The bottom summation gives us

$$\sum_{k=v}^{w-1} \frac{(-1)^k k}{(-k-1)!} = \frac{(-1)^w}{(-w-1)!} - \frac{(-1)^v}{(-v-1)!}$$

for any $v < w \leq 0$, or equivalently

$$\sum_{k=v}^{w-1} \frac{(-1)^k k}{(k-1)!} = \frac{(-1)^{w+1}}{(w-2)!} - \frac{(-1)^{v+1}}{(v-2)!}$$

for any $1 \leq v < w$.

If we consider $\varphi(z)$ in the half-plane $\operatorname{Re} z \geq 0$, then $\operatorname{depth}(\varphi) = \operatorname{depth}(\psi) = 0$, and we have

$$\sum_{k=v}^{w-1} k\Gamma(k+1) = \Gamma(w+1) - \Gamma(v+1)$$

for any $0 \leq v < w$ or, equivalently,

$$\sum_{k=v}^{w-1} k \cdot k! = w! - v!.$$

Example 3 The rational function $\varphi(z) = \frac{1}{z(z+1)}$ satisfies the equation $L(y) = 0$ where $L = (z + 2)E - z$. We have $R = -z - 1$, and $\psi(z) = R(\varphi)(z) = -\frac{1}{z}$. It is easy to see that $\text{depth}(\varphi) = \text{depth}(\psi) = -1$ and

$$\text{bott}(\varphi)(k) = \begin{cases} 1, & \text{if } k = 0, \\ -1, & \text{if } k = -1, \\ 0, & \text{otherwise,} \end{cases}$$

$$\text{bott}(\psi)(k) = \begin{cases} -1, & \text{if } k = 0, \\ 0, & \text{otherwise.} \end{cases}$$

A simple direct check shows that for any $v < w$

$$\sum_{k=v}^{w-1} \text{bott}(\varphi)(k) = \text{bott}(\psi)(w) - \text{bott}(\psi)(v).$$

If we consider $\varphi(z)$ only in the half-plane $\operatorname{Re} z > 0$, then $\operatorname{depth}(\varphi) = \operatorname{depth}(\psi) = 0$, and we have

$$\sum_{k=v}^{w-1} \frac{1}{k(k+1)} = -\frac{1}{w} + \frac{1}{v}$$

for any $1 \leq v < w$.

an example related to the case $\text{ord } L > 1$

Example 4 *For the second-order operator*

$$L = (z - 3)(z - 2)(z + 1)E^2 - (z - 3)(z^2 - 2z - 1)E - (z - 2)^2$$

there exists the first-order summing operator

$$R = zE + \frac{1}{z - 3}.$$

It follows from results of M. Barkatou that the equation $L(y) = 0$ has solutions holomorphic in the half-plane $\text{Re } z > 2$. Denote by $\varphi(z)$ an arbitrary solution of this kind. By our theorem the DNLF must be correct for $3 \leq v < w$ in spite of the fact that one of the coefficients of R has a pole at $z = 3$.

We can find the values of $\varphi(z)$ and of $\psi(z) = R(\varphi)(z)$ when $z = 3$.
An algorithm of Abramov and van Hoeij yields

$$\varphi(z) = (\varphi(4) + 4\varphi(5))(z - 3) + O((z - 3)^2), \quad z \rightarrow 3,$$

that gives $\varphi(3) = 0$, $\psi(3) = 4\varphi(4) + 4\varphi(5)$. This value of $\psi(3)$ can be used in the DNF when $3 = v < w$.

conclusions

1) Indiscriminate application of the discrete Newton-Leibniz formula to the output of Gosper's algorithm or of the Accurate Summation algorithm in order to compute a definite sum can lead to incorrect results. This can be observed in many implementations of these algorithms in computer algebra systems.

2) In this presentation it is shown, in particular, that such undesirable phenomena cannot occur if the elements of the sequence under summation are the values $\varphi(k)$, $k \in \mathbb{Z}$, of an analytic function $\varphi(z)$, which satisfies (in the complex plane \mathbb{C}) the same difference equation with polynomial coefficients as does the original sequence (at integer points).

A practical consequence of this is as follows. If these conditions are satisfied, then a computer-algebra-system user can be sure that the obtained sum was computed correctly.

3) On the more theoretical side, if $\varphi(z)$ mentioned above has some poles at integer points, then nevertheless one can find the sum of a sequence which, however, is not the sequence of values of $\varphi(k)$, $k \in \mathbb{Z}$, but is associated with $\varphi(z)$ in a natural way. This can yield an interesting (and, probably, unexpected) identity. We call this sequence associated with $\varphi(z)$, the *bottom* of $\varphi(z)$. If $\varphi(z)$ is defined for all $z \in \mathbb{Z}$ then its bottom coincides with the sequence $\varphi(k)$, $k \in \mathbb{Z}$.