The Satisfiability Problem: Random Walk and Derandomization

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21. September 2014 - 3. October 2014

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Introduction

- u literal $\Leftrightarrow$ $u{=}x$(variable) or $u = \bar{x}$ (negation of x)
- A finite set C of literals over pairwise distinct variables is called a clause.
- A finite set of clauses is called a formula in CNF (Conjunctive Normal Form). CNFs which have no more than k distinguishing literals are denoted ($\leq k$) CNF formulas, ones with exactly k literals k-CNF formulas.

Example

For a ($\leq$ 3)-CNF formula

$$(\bar{a} \lor \bar{b} \lor \bar{c}) \land (a \lor c) \qquad\qquad a, b, c \in \{0, 1\}$$

Introduction

- The task of deciding whether a CNF-formula F is satisfiable is labelled satisfiability problem (short: SAT)
- A mapping α: V $\rightarrow$ $\{$0,1$\}$ is called (truth) assignment.
- SAT is first NP-complete problem (shown by Stephen A.Cook)

brute-force and splitting

- assignment F is satisfiable iff $F|_{[x:=1]}$ or $F|_{[x:=0]}$ is
  $\Rightarrow$ recursive $O(2^n poly(n))$-algorithm
  $\Rightarrow$ upper bound for the runtime
- Especially for small k better runtimes possible: for instance for 3-SAT Rodošek achieved $O(1.476^n)$

Random Walk algorithm by Uwe Schöning

The basic idea

- applying a Monte Carlo approach onto k-SAT
- similarities:
  - same configuration space $\mathbb{Z}_2^n$
  - Choosing an initial element uniformly at random
- differences
  - Selection of the coordinate/atom/literal
  - "Flipping rule"

The basic idea

The probability that we do not find a satisfying assignment after t repetitions with independent random bits is

$$[1 - \mathbf{Pr}(N \leq 3n)]^t \leq e^{-\mathbf{Pr}(N \leq 3n)t}$$

Example

For an error probability less than 10^{-3}, you need $t := \frac{3 \cdot ln(10)}{\mathbf{Pr}(N \leq 3n)}$ independent repetitions of Schöning's algorithm.

Pseudocode

Schöning(($\leq$ k)-CNF formula F)

$\alpha \overset{\text{u.a.r.}}{\leftarrow} \{0,1\}^n$ // sample α **u**niformly **a**t **r**andom
return Schöning-Walk(F, α)

Schöning-Walk(($\leq$ k)-CNF formula F, assignment α)

for i = 0, .., t do
for i = 0, .., 3n do
if α satisfies F **then**
return α
else
C $\leftarrow$ any clause of F unsatisfied by α
u $\overset{\text{u.a.r.}}{\leftarrow}$ C // a random literal from C
$\alpha \leftarrow \alpha[u \mapsto 1]$
endif
endfor
return failure
endfor

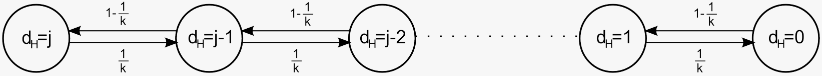
Estimate of the runtime

Theorem 1

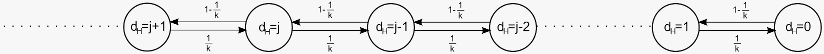
If the input k-SAT instance is satisfiably, then algorithm Schöning needs on expectation $O\left(\left(\frac{2(k-1)}{k}\right)^n\right)$ tries to find a satisfying assignment.

Proof

- WANTED : $\mathbf{Pr}(\exists t \leq 3n : Y_t = 0)$



- Consider the Markov chain:



- Let N_j the random variable that counts the number of steps until the first encounter of state 0

Estimate of the runtime

Lemma 2.2

For $q < \frac{1}{2}$ and $j \in \mathbb{N}_0$:

$$\mathbf{Pr}(N_j < \infty) = \left(\frac{q}{1-q}\right)^j$$

Proof

$$\begin{aligned}\mathbf{Pr}(N_j < \infty) &= \sum_{i=0}^{\infty} \binom{2i+j}{i} \cdot \frac{j}{2i+j} \cdot (1-q)^i \cdot q^{i+j} \\ &= q^j \cdot \sum_{i=0}^{\infty} \binom{2i+j}{i} \cdot \frac{j}{2i+j} \cdot (q \cdot (1-q))^i \\ &= q^j \cdot (B_2(q(1-q)))^j\end{aligned}$$

Estimate of the runtime

for $B_2(z)$ being the generalized Binomial series defined by

$$B_2(z) = \sum_i \binom{2i+1}{i} \cdot \frac{z^i}{2i+1} = \frac{1-\sqrt{1-4z}}{2z}$$

for which

$$(B_2(z))^r = \sum_i \binom{2i+r}{i} \cdot \frac{r}{2i+1} \cdot z^i$$

$\forall r \in \mathbb{N}_0$. So

$$\mathbf{Pr}(N_j < \infty) = q^j \cdot \left(\frac{1-\sqrt{1-4q+4q^2}}{2(1-q)q}\right)^j = q^j \cdot \left(\frac{1}{1-q}\right)^j$$

Estimate of the runtime

Lemma 2.3

For $q < \frac{1}{2}$ andl $j \in \mathbb{N}_0$ it holds:

$$\mathbf{E}(N_j \mid N_j > \infty) = \frac{j}{1-2q}$$

Proof

$$\mathbf{E}(N_j \mid N_j < \infty) = \frac{1}{\mathbf{Pr}(N_j < \infty)} \cdot \sum_{i=0}^{\infty} (2i+j) \cdot \binom{2i+j}{i} \cdot \frac{j}{2i+j} \cdot (1-q)^i \cdot q^{i+j}$$

$$\overset{\text{Lemma2.2}}{=} j \cdot (1-q)^j \cdot \sum_{i=0}^{\infty} (2i+j) \cdot \binom{2i+j}{i} \cdot (q \cdot (1-q))^i$$

$$= j \cdot (1-q)^j \cdot \frac{(B_2(q(1-q)))^j}{\sqrt{1-4q(1-q)}} = \frac{j}{1-2q}$$

Estimate of the runtime

Lemma 2.4

Let N the random variable that counts the number of steps until state 0 is encountered for the first time. For $q < \frac{1}{2}$ it holds, while n is still the number of variables:

$$\mathbf{Pr}(N < \infty) = \left(\frac{1}{2(1-q)}\right)^n$$

Proof

$$\mathbf{Pr}(N < \infty) = \sum_{j=0}^{n} \binom{n}{j} \cdot 2^{-n} \cdot \mathbf{Pr}(N_j < \infty)$$

$$\overset{\text{Lemma2.2}}{=} \sum_{j=0}^{n} \binom{n}{j} \cdot 2^{-n} \cdot \left(\frac{q}{1-q}\right)^j = \left(\frac{1}{2(1-q)}\right)^n$$

Estimate of the runtime

Lemma 2.5

For $q < \frac{1}{2}$ it holds:

$$\mathbf{E}(N \mid N < \infty) = \frac{qn}{1-2q}$$

Proof

$$\begin{aligned}
\mathbf{E}(N \mid N < \infty) &= \sum_i i \cdot \mathbf{Pr}(N = i \mid N < \infty) = \\
&= \sum_i i \cdot \sum_{j=0}^{n} \binom{n}{j} \cdot \mathbf{Pr}(N_j = i \mid N < \infty) = \\
&= \frac{2^{-n}}{\mathbf{Pr}(N < \infty)} \cdot \sum_{j=0}^{n} \binom{n}{j} \cdot \sum_i i \cdot \mathbf{Pr}(N_j = i) =
\end{aligned}$$

Estimate of the runtime

$$= \frac{2^{-n}}{\mathbf{Pr}(N < \infty)} \cdot \sum_{j=0}^{n} \binom{n}{j} \cdot \mathbf{E}(N_j \mid N_j < \infty) \cdot \mathbf{Pr}(N_j < \infty)$$

$$\stackrel{\text{Lemma2.2,2.3,2.4}}{=} (1-q)^n \cdot \sum_{j=0}^{n} \binom{n}{j} \cdot \frac{j}{1-2q} \cdot \left(\frac{q}{1-q}\right)^j$$

$$= \frac{n \cdot (1-q)^n}{1-2q} \cdot \sum_{j=0}^{n} \binom{n-1}{j-1} \cdot \left(\frac{q}{1-q}\right)^j$$

$$= \frac{n \cdot (1-q)^n}{1-2q} \cdot \frac{q}{1-q} \cdot \left(1 + \frac{q}{1-q}\right)^{n-1} = \frac{nq}{1-2q}$$

Estimate of the runtime

Lemma 2.6

For $q < \frac{1}{2}$ and $\lambda \geq 1$ it holds:

$$\mathbf{Pr}\left(N \leq \frac{\lambda q n}{1-2q}\right) > \left(1 - \frac{1}{\lambda}\right) \cdot \left(\frac{1}{2(1-q)}\right)^{n}$$

Proof:
Write μ for $\mathbf{E}(N \mid N < \infty)$.Observe that

$$\mathbf{Pr}(N > \lambda\mu \mid N < \infty) < \frac{1}{\lambda}$$

by Markov's inequality, and
$\mathbf{Pr}(N \leq \lambda\mu) = \mathbf{Pr}(N > \lambda\mu \mid N < \infty) \cdot \mathbf{Pr}(N < \infty)$

Estimate of the runtime

Now using $q = \frac{1}{k}, k \geq 3$ and $\lambda = 3$, we obtain

$$\mathbf{Pr}(\exists t \leq 3n : Y_t = 0) = \mathbf{Pr}(N \leq 3n) > \frac{2}{3}\left(\frac{k}{2(k-1)}\right)^n$$

whereas, for a satisfiable formula, the expected number of repetitions of procedure Schöning until a satisfying assignment is found is at most $\frac{1}{\mathbf{Pr}(N \leq 3n)}$:

$$\frac{3}{2}\left(\frac{2(k-1)}{k}\right)^n$$

Derandomization attempt by Dantsin et al.

The basic idea

- $d_H(\alpha, \beta) := |\{x \in V | \alpha(x) \neq \beta(x)\}|$ with α, β truth assignments is called Hamming distance
- $B_r(\alpha) := \{d_H(\alpha, \beta) \leq r\}$ is denoted Hamming ball, with volume $vol(n, r) := |B_\alpha(r)| = \sum_{i=0}^{r} \binom{n}{i}$.
- $C \subseteq \{0, 1\}^n$ covering code of radius r and length n $\Leftrightarrow \cup_{\alpha \in C} B_r(\alpha) = \{0, 1\}^n$
- Ball-k-SAT: Decide whether $B_r(\alpha)$ contains a satisfying assignment

Pseudocode

cover-search(($\leq$ k)-CNF formula F over n variables)

$r := \frac{n}{k+1}$
construct a covering code C of radius r and $|C| \leq \frac{2^n}{\binom{n}{k}} poly(n)$
return $\bigcup_{\alpha \in C} sat - searchball(F, \alpha, r)$

$sat - searchball((\leq k)$-CNF formula F, assignment α, radius r)

if α satisfies F **then**
    return true
elseif r = 0 **then**
    return false
else
    C $\leftarrow$ any clause of F unsatisfied by α
    return $\bigcup_{u \in C} sat - searchball(F|_{u:=1}, \alpha, r - 1)$
endif

Correctness of this algorithm

Initial step

- $r = 0 \Rightarrow B_0(\alpha) = \{\alpha\}$

Induction step $(r - 1 \rightarrow r)$

- Let be α^* a satisfying assignment with $d_H(\alpha, \alpha^*) \leq r$ and C the selected clause
- Let $\alpha' := \alpha^*[u := 0]$
- We observe that $d_H(\alpha, \alpha') \leq r - 1$ and α' satisfies $F|_{u:=1}$(induction hypothesis).

Estimate of the runtime

Lemma 2.12

The algorithm sat-searchball solves BALL-k-SAT in time $O(k^r poly(n))$.

Proof:

If F is a $(\leq k)$-CNF formula, then each call to searchball causes at most k recursive calls.

Estimate of the runtime

Theorem 2

Suppose some algorithm A solves BALL-k-SAT in O(a^rpoly(n)) steps. Then there is an algorithm B solving k-SAT in time $O\left(\left(\frac{2a}{(a+1)}\right)^n poly(n)\right)$, and B is deterministic if A is.

Proof

Lemma 2.13

For all n $\in \mathbb{N}$, $0 \leq r \leq n$, every code C of covering radius r and length n has at least $\frac{2^n}{vol(n,r)}$ elements. Furthermore, there is such a C with

$$|C| \leq \frac{2^n poly(n)}{vol(n,r)},$$

and furthermore, C can be constructed deterministically in time O($|C|$ poly(n)).

Proof: This Lemma is the case k=2 of upcoming Lemma 2.19.

Estimate of the runtime

Lemma 2.14

For $0 \leq \rho \leq \frac{1}{2}$ and t $\in \mathbb{N}$, it holds that

$$\binom{t}{\rho t} \geq \frac{1}{\sqrt{8t\rho(1-\rho)}} \left(\frac{1}{\rho}\right)^{\rho t} \left(\frac{1}{1-\rho}\right)^{(1-\rho)t}$$

Set r $:= \frac{n}{(a+1)}$ and construct a covering code C of radius r and length n and call A(F, α, r) for each $\alpha \in$ C.

$$|C| a^r poly(n) \overset{\text{Lemma2.13}}{\leq} \frac{2^n a^r poly(n)}{vol(n,r)} \overset{\text{Lemma2.14}}{\leq} \frac{2^n a^{\frac{n}{a+1}} poly(n)}{(a+1)^{\frac{n}{a+1}} \left(\frac{a+1}{a}\right)^{\frac{na}{a+1}}} =$$

$$= \left(\frac{2a}{a+1}\right)^n poly(n)$$

Complete Derandomization by Moser & Scheder

The basic idea

- further development of Dantsin et al.'s covering code idea
- Promise-Ball-k-SAT:
  Ball-k-SAT+ promise that $B_r(\alpha)$ contains a satisfying assignment
- k truth values instead of 2
  - $vol^{(k)}(t,r) := |B^{(k)}(w)| = \binom{t}{r}(k-1)^r$, w $\in \{1,...,k\}^t$
  - Let t $\in \mathbb{N}$. A set C$\subseteq \{1,..,k\}^t$ is called k-ary covering code of radius r $\Leftrightarrow \bigcup_{w\in C} B_r^{(k)}(w) = \{1,..,k\}^t$

Pseudocode

searchball − fast($k \in \mathbb{N}$, ($\leq$ k)-CNF formula F, assignment α, radius r,
code C$\subseteq \{1, ..., k\}^t$)

if α satisfies F **then**
    return true
elseif r $= 0$ **then**
    return false
else
    G $\leftarrow$ a maximal set of pairwise disjoint k-clauses of F unsatisfied by α
    if $|G| < t$ **then**
        return $\bigcup_{\beta \in \{0,1\}^{vbl(G)}} sat - searchball(F|_{[\beta]}, \alpha, r)$
    else
        H $\leftarrow \{C_1, ..., C_t\} \subseteq$ G
        return $\bigcup_{w \in C} searchball - fast(k, F, \alpha[H, w], r - (t - 2t/k), C)$
    endif
endif

Estimate of the runtime

Lemma 2.19

$\forall t, k \in \mathbb{N}$ and $0 \leq s \leq \frac{t}{2}$. $\exists$ C $\subseteq \{1, ..., k\}^t$ of covering radius s such that:

$$|C| \leq \left\lceil \frac{\ln(k) \cdot k^t \cdot t}{\binom{t}{s} \cdot (k-1)^s} \right\rceil =: m$$

and furthermore, C can be constructed deterministically in time O($|C|$ poly(t)).

Proof:

$$\mathbf{Pr}\left[w' \notin \bigcup_{w \in C} B_s^{(k)}(w)\right] = \left(1 - \frac{vol^{(k)}(t,s)}{k^t}\right)^{|C|} < e^{-|C| \frac{vol^{(k)}(t,s)}{k^t}}$$

$$\leq e^{-t \cdot \ln(k)} = k^{-t}$$

Estimate of the runtime

runtime of constructing such a covering code

- t:= $\lfloor \ln(n) \rfloor$

$$O(|C| \cdot poly(t)) \leq O(k^t \cdot poly(t)) = O(n^{\ln(k)} \cdot ploy(\ln(n))) = O(poly(n))$$

Estimate of the runtime

Lemma 2.20

With the aid of Lemma 2.14 and Lemma 2.19 we get the following approximation of $|C|$ which we will need later. Let ρ be $\frac{1}{k}$:

$$\binom{t}{\frac{t}{k}} \geq \frac{1}{\sqrt{8}} \cdot k^{\frac{t}{k}} \cdot \left(\frac{k}{k-1}\right)^{\frac{(k-1)\cdot t}{k}} = \frac{k^t}{\sqrt{8t} \cdot (k-1)^{\frac{(k-1)t}{k}}}$$

So we obtain, for t sufficiently large:

$$|C| \leq \left\lceil \frac{\ln(k) \cdot k^t \cdot t}{\binom{t}{s} \cdot (k-1)^s} \right\rceil \leq \frac{t^2 \cdot k^t \cdot (k-1)^{\frac{(k-1)t}{k}}}{k^t \cdot (k-1)^{\frac{t}{k}}} = t^2 \cdot (k-1)^{t - \frac{2t}{k}}$$

Estimate of the runtime

Case m $<$ t

```
6:  G ← a maximal set of pairwise disjoint k-clauses of F unsatisfied by α
7:  if |G| < t then
8:      return ∨_{β∈{0,1}^{vbl(G)}} sat-searchball(F|_[β], α, r)
```

Lemma 2.21

If every clause in F that is not satisfied by α has size at most k - 1, then sat-searchball(F, α, r) runs in time O($(k-1)^r$poly(n)).

Proof:

Since every unsatisfied clause and every $F|_{u:=1}$ hade at most k - 1 literals, the algorithm calls itself at worst k – 1 times.

Estimate of the runtime

Case m < t

$$2^{km}O((k-1)^r poly(n)) \leq O(2^{kt}(k-1)^r poly(n)) \leq$$
$$\leq O(2^{ln(n)k}(k-1)^r poly(n)) \leq O((k-1)^r poly(n))$$

$\overset{\text{Theorem 2}}{\Rightarrow}$ runtime k-SAT: $O\left(\left(\frac{2(k-1)}{k}\right)^n\right)$

Estimate of the runtime

Case m $\geq$ t

```
10:          H ← {C_1, ..., C_t} ⊆ G
11:          return ∨_{w∈C} searchball-fast(k,F,α[H,w], r-(t-2t/k),C)
```

- Let α[H,w] be the assignment obtained from α by flipping the value of the w_i^{th} literal
- Define w* $\in \{1, \ldots, k\}^t$ as follows: for each 1$\leq$ i $\leq$t set w_i^* to j such that α^* satisfies the jth literal in C_i.

Observe:

- There is some $w \in \{1, ..., k\}^t$ such that $d_H(\alpha[w^*], \alpha^*) = d_H(\alpha, \alpha^*) - t$
- Let w, w′ $\in \{1, ..., k\}^t$. Then $d_H(\alpha[w], \alpha[w\prime]) = 2d_H(w, w\prime)$

Estimate of the runtime

Case m $\geq$ t

Lemma 2.24

If α^* is a satisfying assignment of F, then there is some w $\in$ C such that $d_H(\alpha[w], \alpha^*) \leq d_H(\alpha, \alpha^*)$ - t + 2s. In particular, if $B_r(\alpha)$ contains a satisfying assignment, then there is some w $\in$ C such that $B_{r-t+2s}(\alpha[w])$ contains it, too.

Proof:

C has covering radius s $\Rightarrow d_H(w, w^*) \leq s$

$\overset{\text{Observation 2}}{\Rightarrow} \; d_H(\alpha[w], \alpha[w^*]) \leq 2s$

$$d_H(\alpha[w], \alpha^*) \leq d_H(\alpha[w^*], \alpha^*) + d_H(\alpha[w], \alpha[w^*]) \leq$$
$$\leq d_H(\alpha, \alpha^*) - t + 2s$$

Estimate of the runtime

Case m $\geq$ t

- $|C|$ recursive calls
- decrease of the complexity parameter r by $t - 2s = t - \frac{2t}{k}$ in each step

$$|C|^{\frac{r}{t-\frac{2t}{k}}} \overset{\text{Lemma 2.20}}{\leq} \left(t^2 \cdot (k-1)^{t-\frac{2t}{k}}\right)^{\frac{r}{t-\frac{2t}{k}}} = \left(t^{\frac{2}{t-\frac{2t}{k}}} \cdot (k-1)\right)^r$$

$$= (k-1)^{r+o(n)}$$

$\overset{\text{Theorem 2}}{\Rightarrow}$ runtime k-SAT: $\mathrm{O}\left(\left(\frac{2(k-1)}{k}\right)^{n+o(n)}\right)$

Theorem 3

There is a deterministic algorithm solving k-SAT in time $O\left(\frac{2(k-1)}{k}\right)^{n+o(n)}$

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