Enumerating labeled trees

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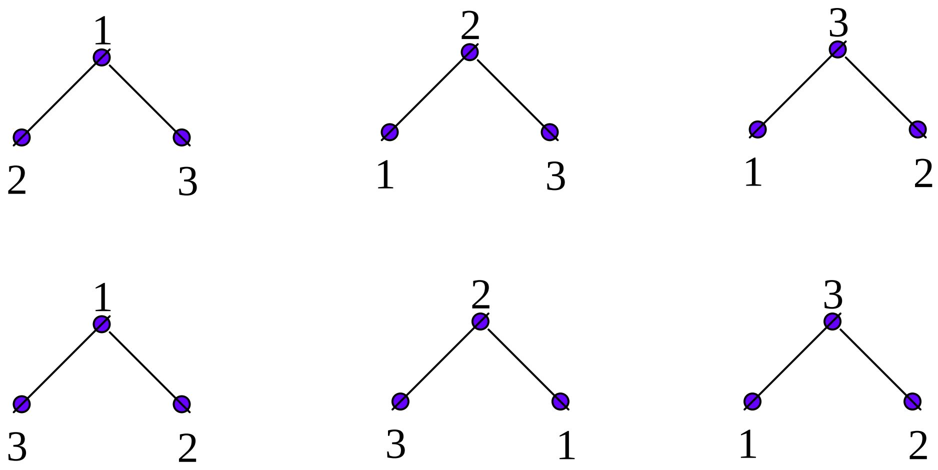

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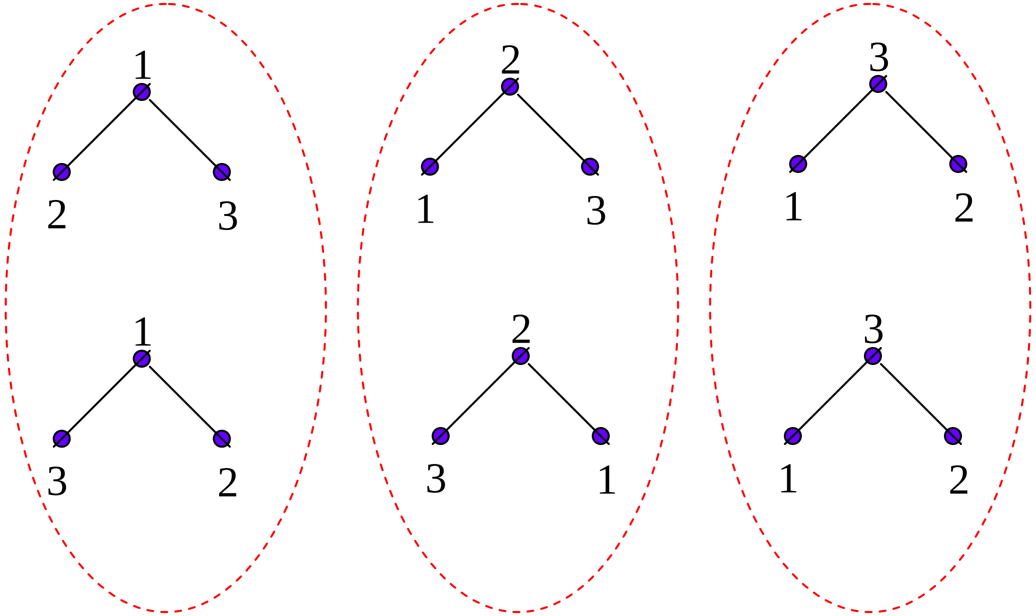

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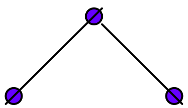

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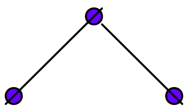

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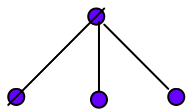

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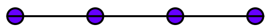

4!/2=12

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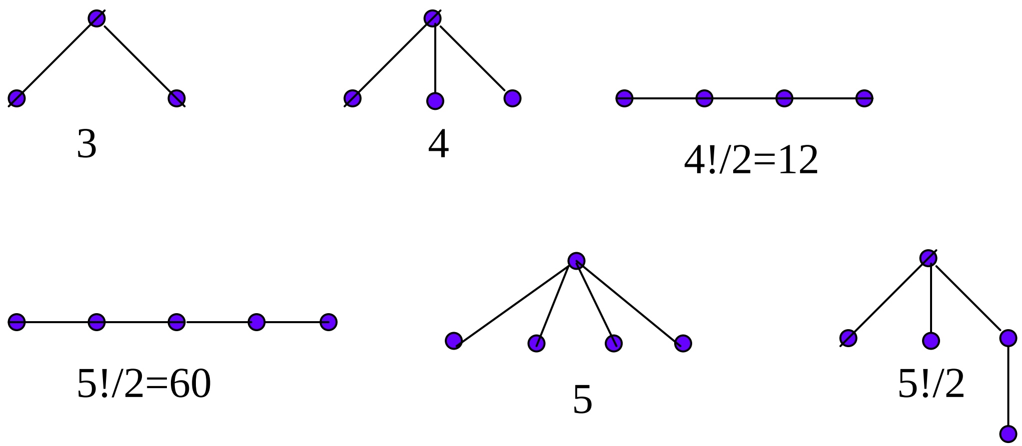

So what is the general formula?

Let T_n denote the number of labeled trees on n vertices. We now know the following values for small n:

$T_2 = 1$ as there is only one tree on 2 vertices

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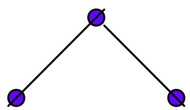

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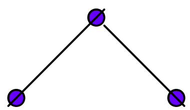
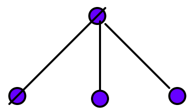


3

$T_4 = 4+12=16$ as we have also seen before:



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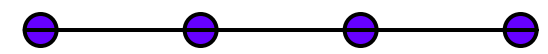

So what is the general formula?

Let T_n denote the number of labeled trees on n vertices. We now know the following values for small n:

T_2 = 1 as there is also one tree on 2 vertices

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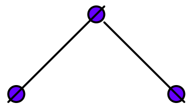

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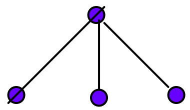

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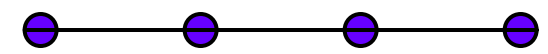
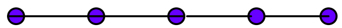
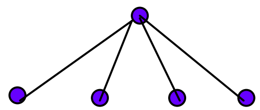


24!/2=12

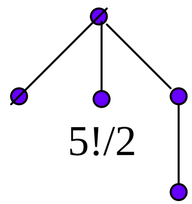
T_5 = 60+5+60=125



5!/2=60



5



5!/2

So what is the general formula?

If we continue in this fashion, we will obtain the following sequence:

1, 3, 16, 125,1296,16807,262144...

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$$T_n = n^{n-2}$$

Cayley's theorem

<u>**Theorem (Cayley)**</u> There are n^{n-2} labeled trees on n vertices.

<u>1. Induction</u>

$A \subset \{1,2,...n\}, |A| = k$

F(A, n) - the set of forests on n vertices in which vertices from A appear in different connected components(trees).

$T_{n,k}$ - the number of forests of k trees, for which the vertices from A appear in different components.

Cayley's theorem - induction

$$A = \{n-k+1, n-k+2, \ldots n\}, |A| = k$$

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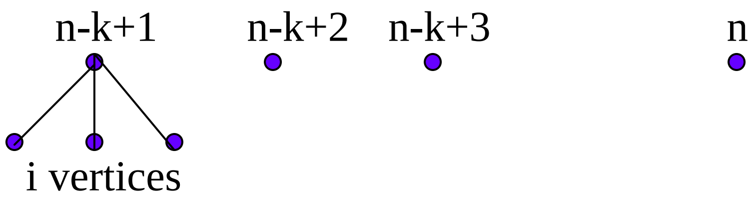
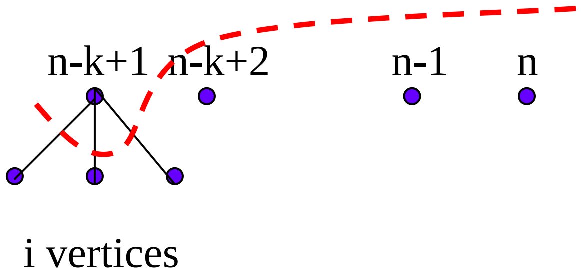
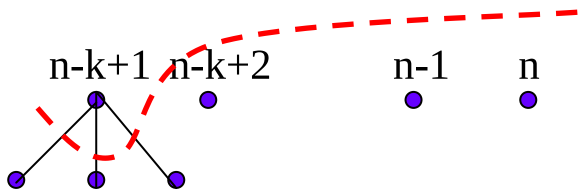
$T_{n,k}$ - the number of forests of k trees, for which the vertices from A appear in different components.




n-k+1
n-k+2
n-1
n
i vertices

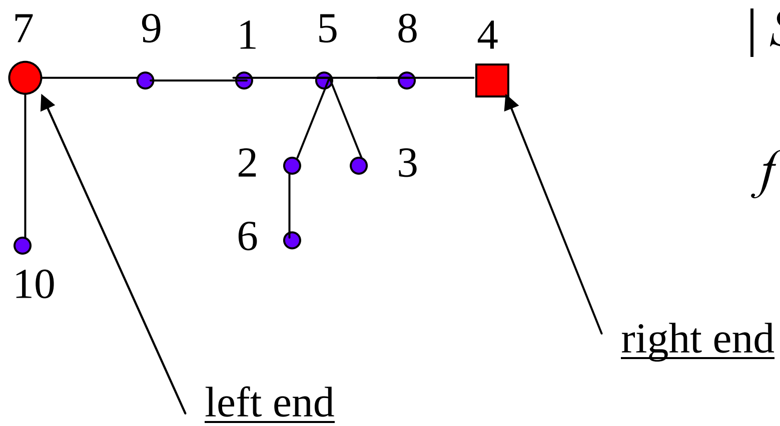


$$F(A,n) \leftrightarrow \{F(A',n-1), A' = (A \setminus \{n\}) \cup \{i \quad chosen \quad vertices\}\}$$

$$T_{n,k} = \sum_{i=0}^{n-k} \binom{(n-1)-(k-1)}{i} T_{n-1,k+i-1}$$

$$T_{n,k} = kn^{n-k-1}$$

2. Establishing a 1-1 correspondence between trees and functions acting from 1..n into 1..n (Joyal)



$$|S_n| = n^2 |T_n|$$

$$f : \{1, \ldots, n\} \to \{1, \ldots, n\}$$

$$f|_M = \begin{pmatrix} 1 & 4 & 5 & 7 & 8 & 9 \\ 7 & 9 & 1 & 5 & 8 & 4 \end{pmatrix}$$

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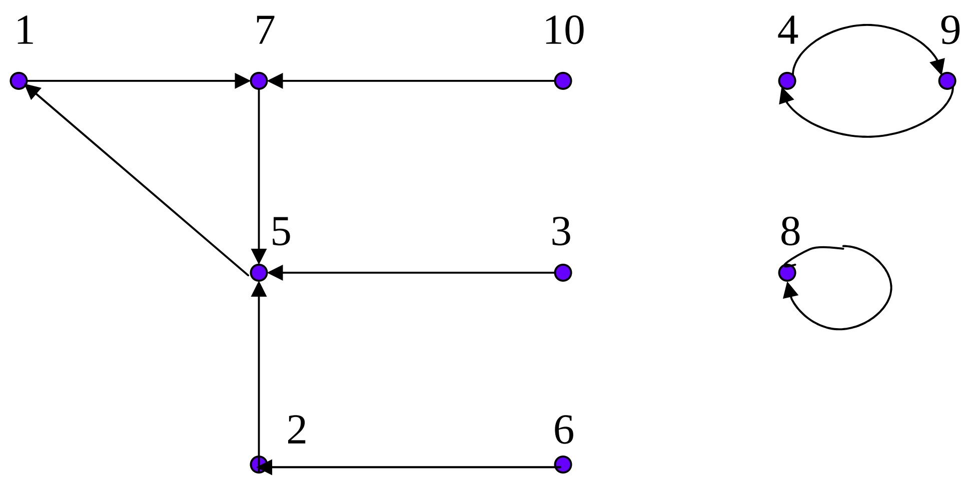
$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 7 & 5 & 5 & 9 & 1 & 2 & 5 & 8 & 4 & 7 \end{pmatrix}$$



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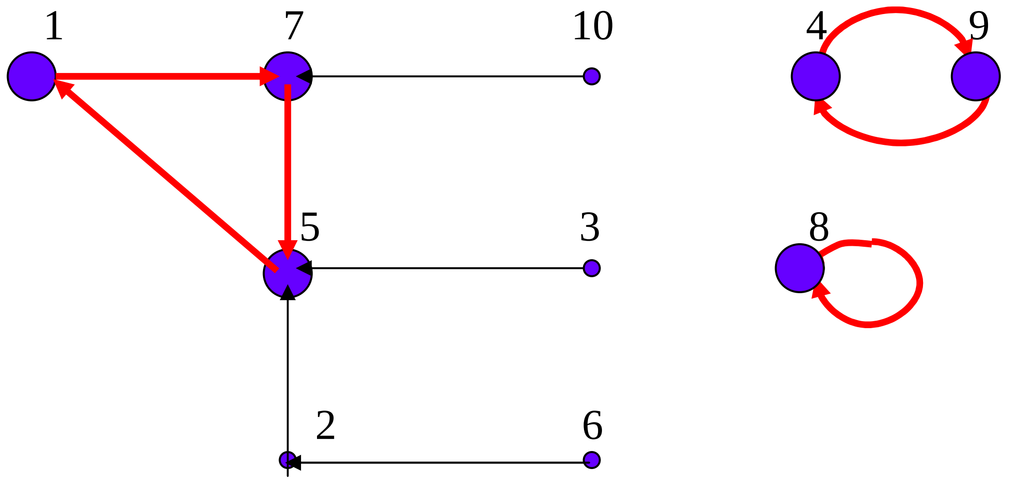

$M = \{1,4,5,7,8,9\}$ $f|_M$ is a bijection

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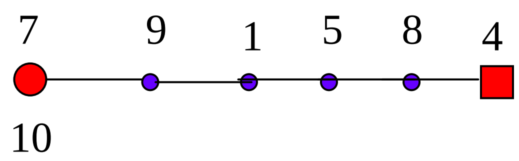

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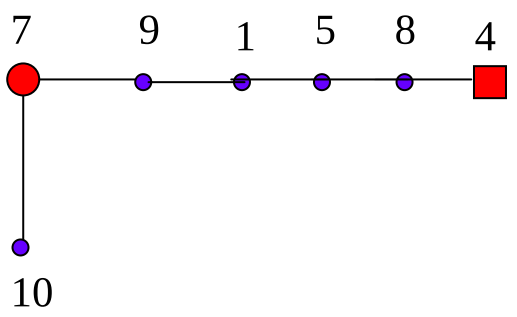

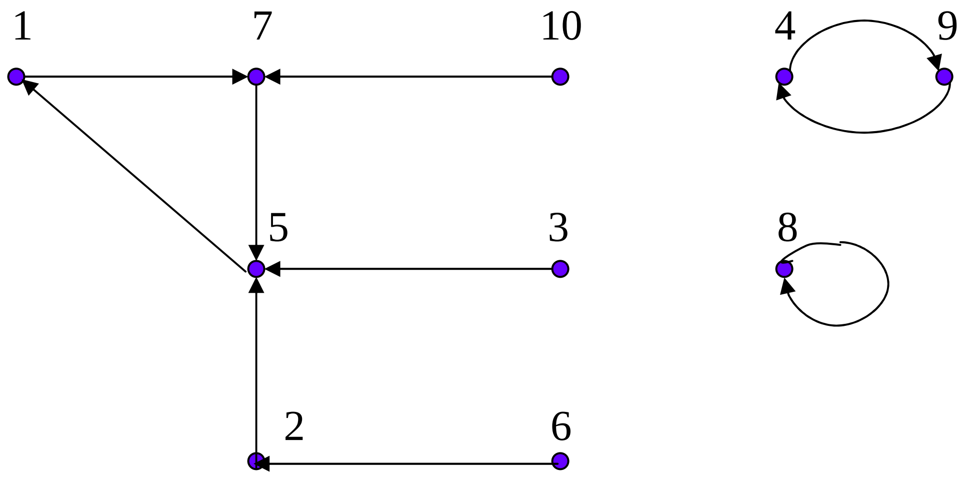

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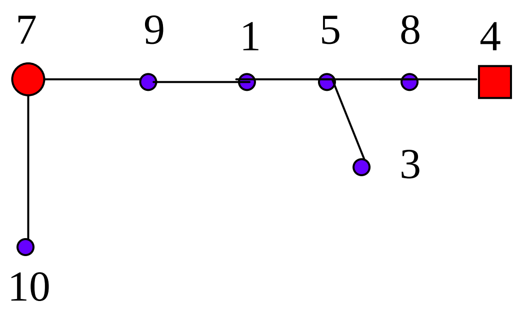

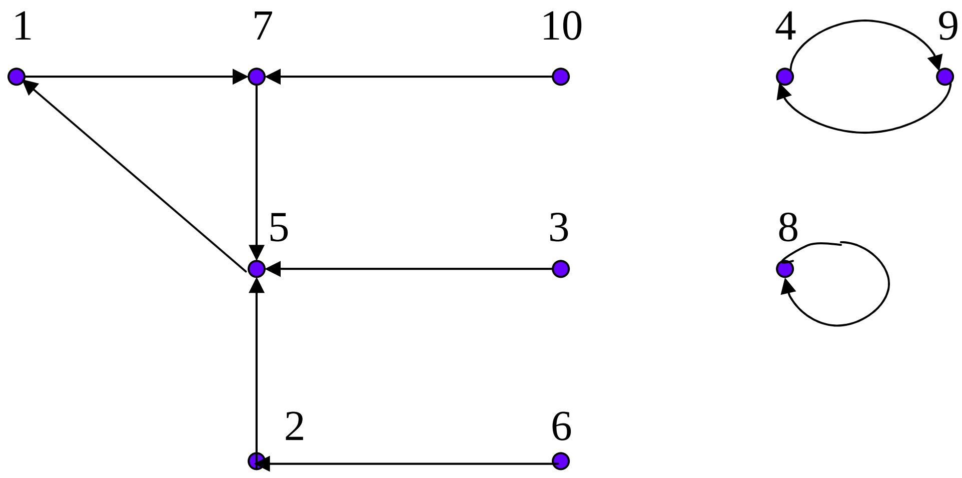

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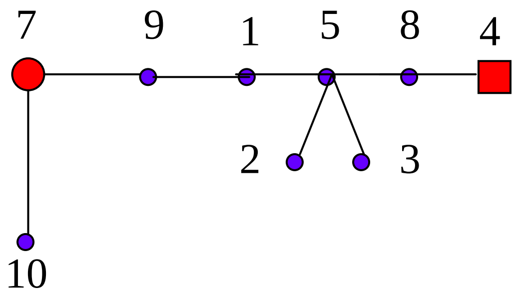

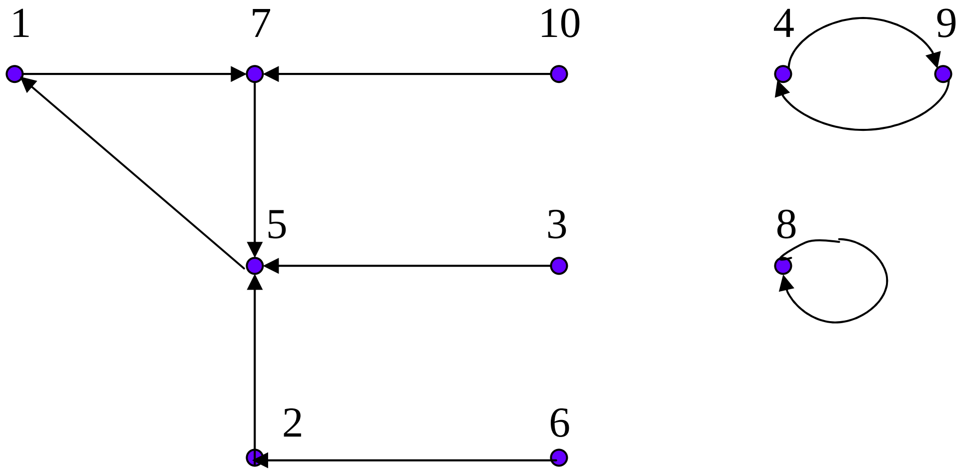

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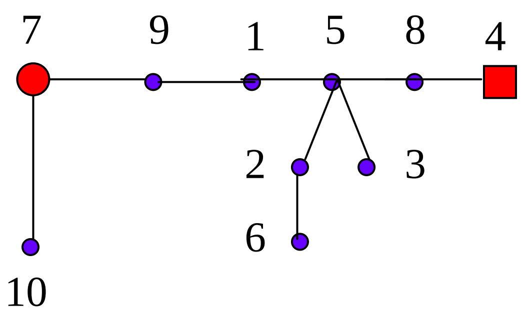

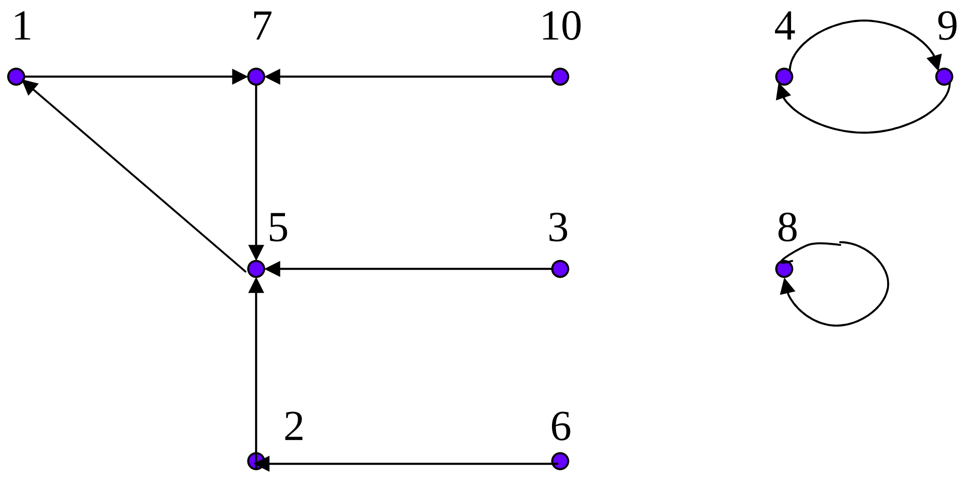

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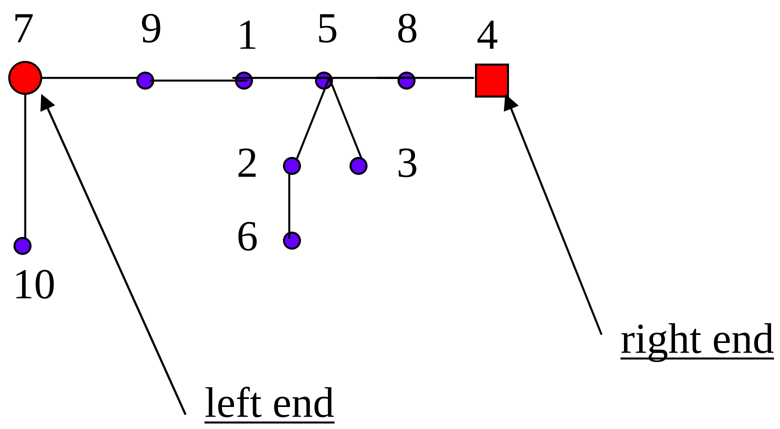

(7,9,1,5,8,4) -> (1,5,7,8,4,9)

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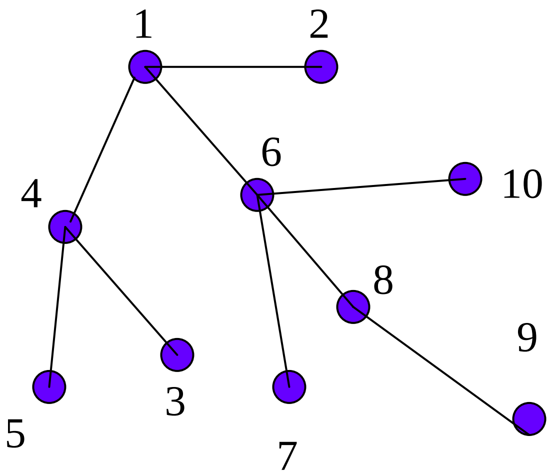
2. Pruefer code



Labeled tree -> $(a_1, a_2, \ldots, a_{n-2})$

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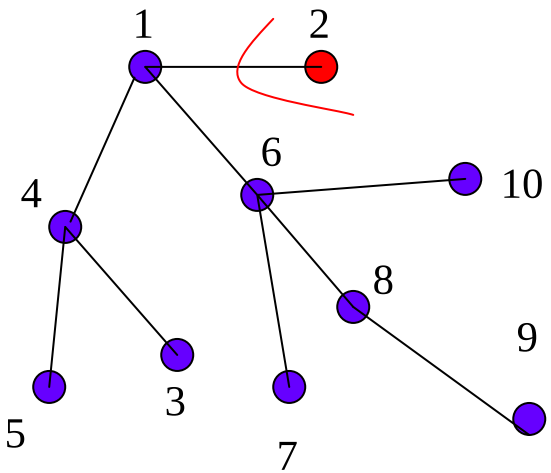

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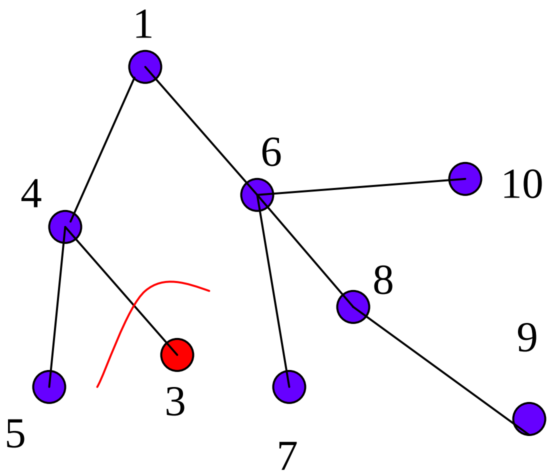
Pruefer code :1

2. Pruefer code



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Pruefer code :1 4

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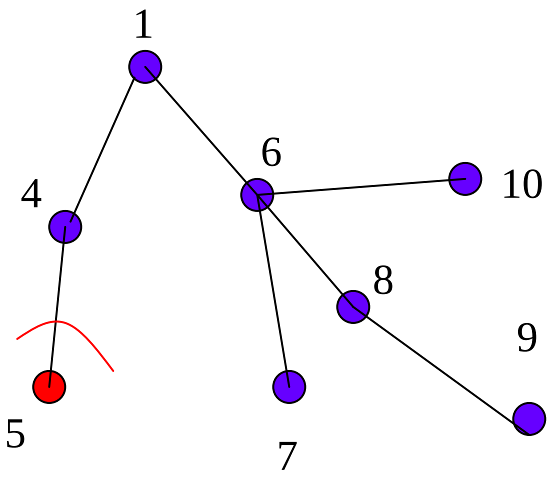

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Pruefer code :1 4 4

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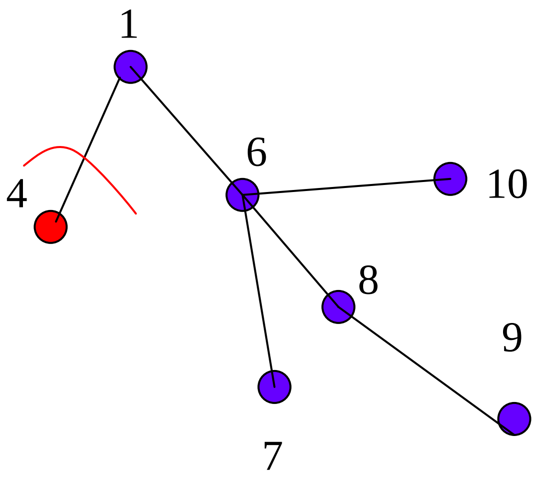

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Pruefer code :1 4 4 1

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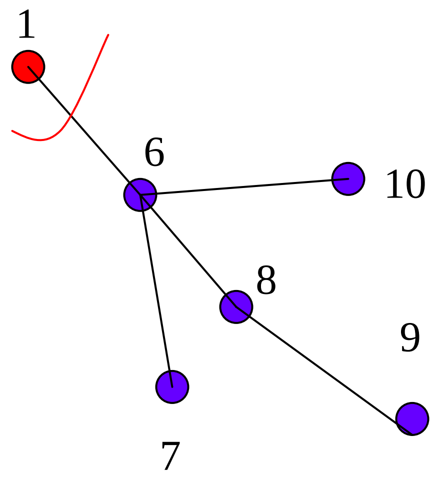

Labeled tree -> $(a_1, a_2, \ldots, a_{n-2})$

Pruefer code :1 4 4 1 6

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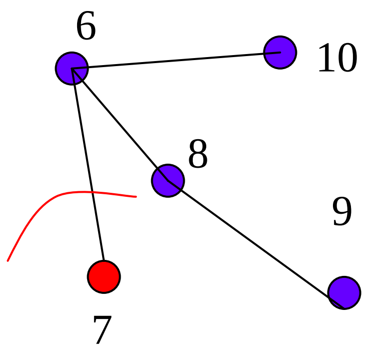

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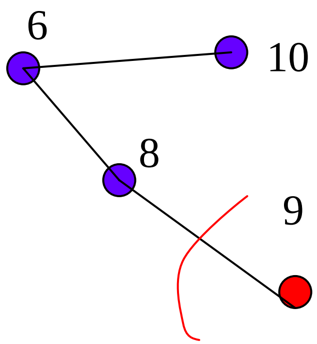

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Pruefer code :1 4 4 1 6 6 8

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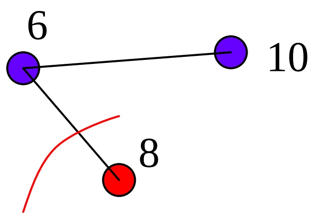

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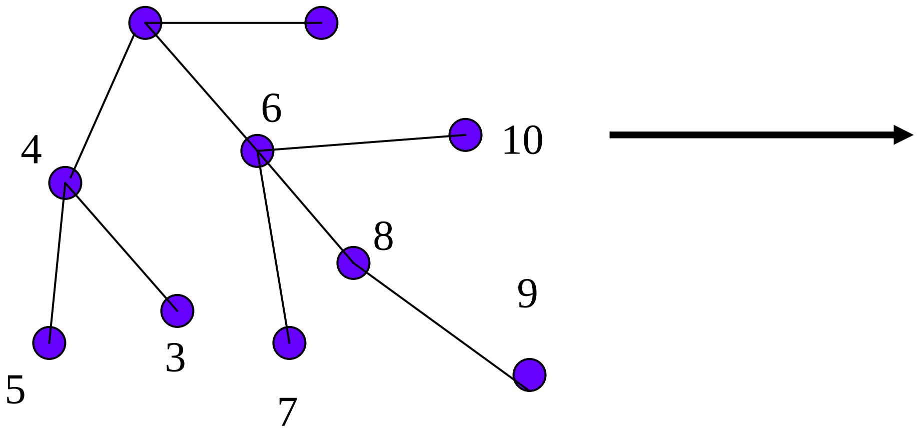

(1 4 4 1 6 6 8 6)

Reversing the correspondence

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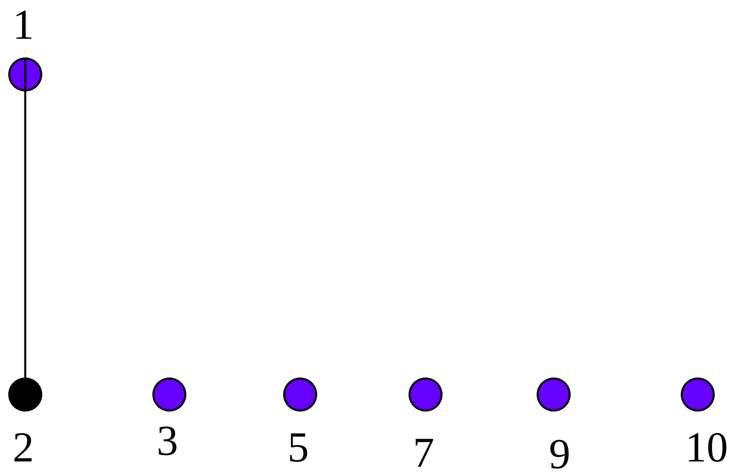

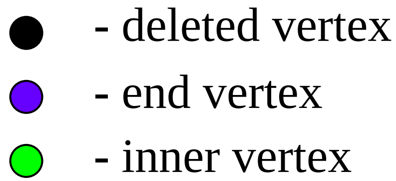

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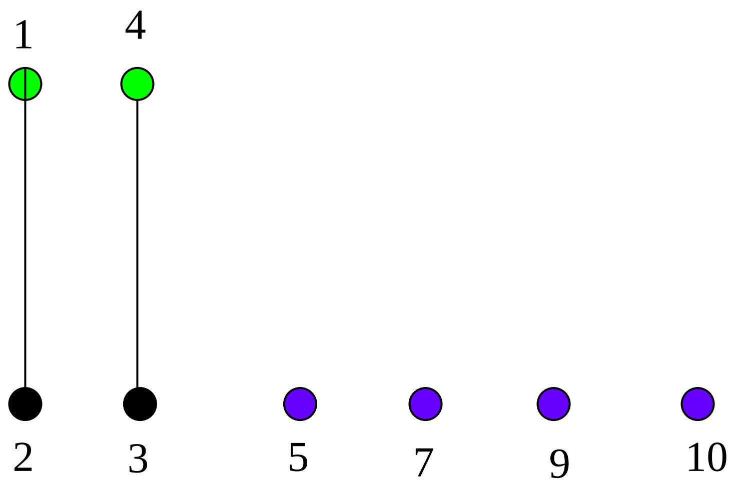

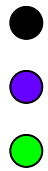

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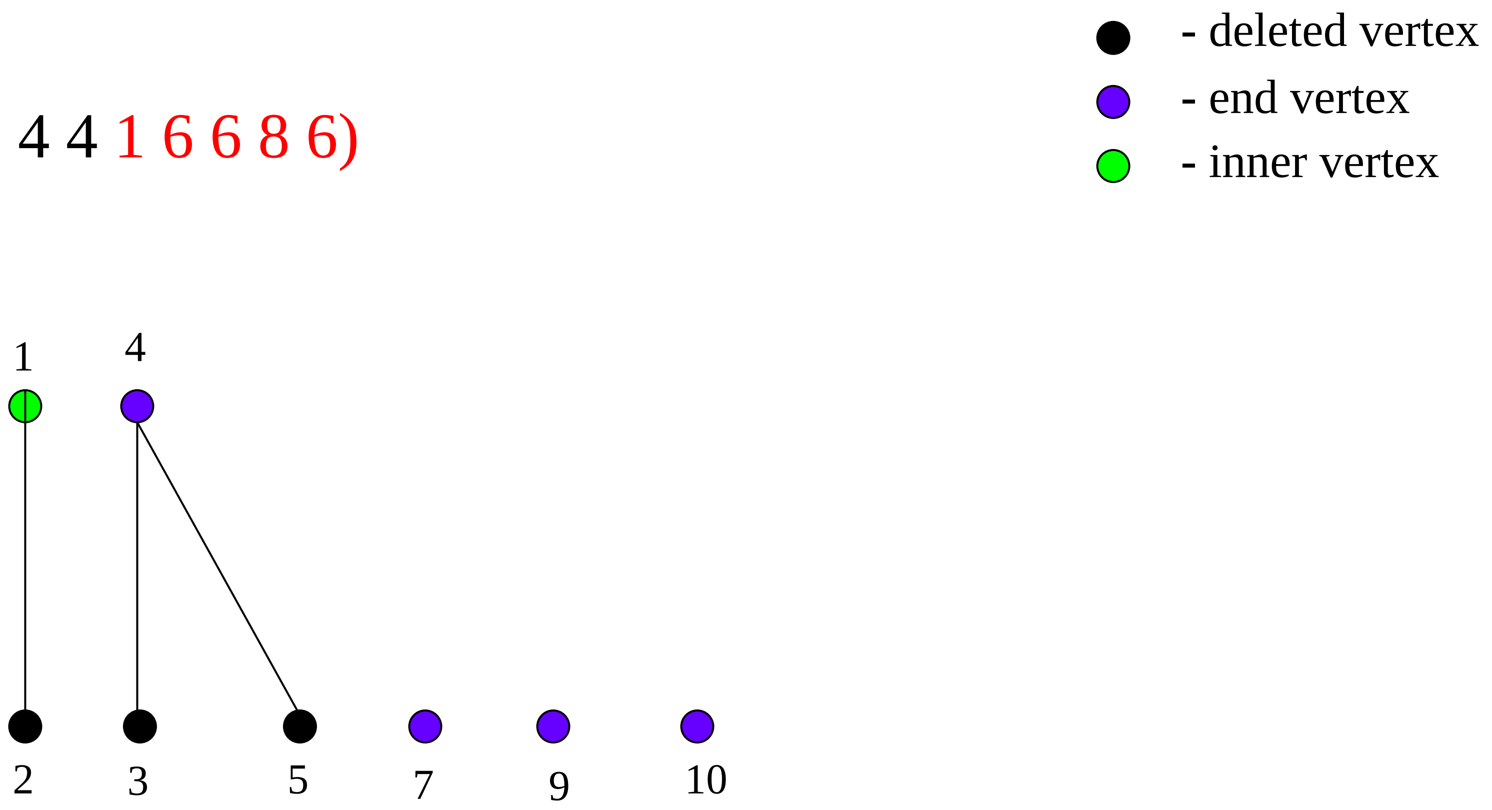

Reversing the correspondence

- deleted vertex
- end vertex
- inner vertex

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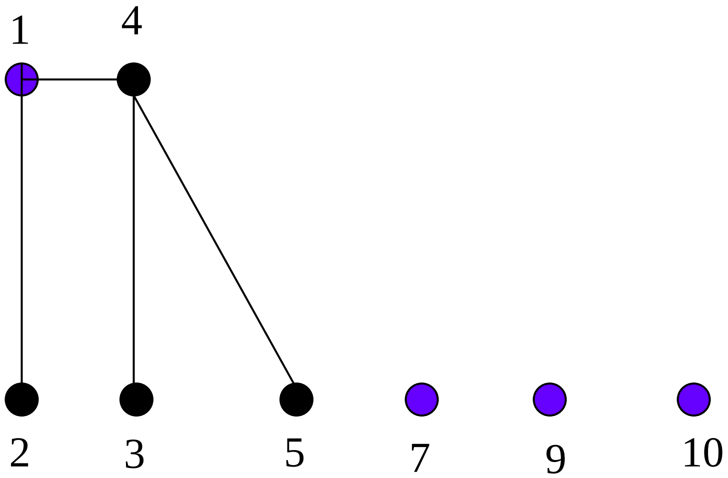

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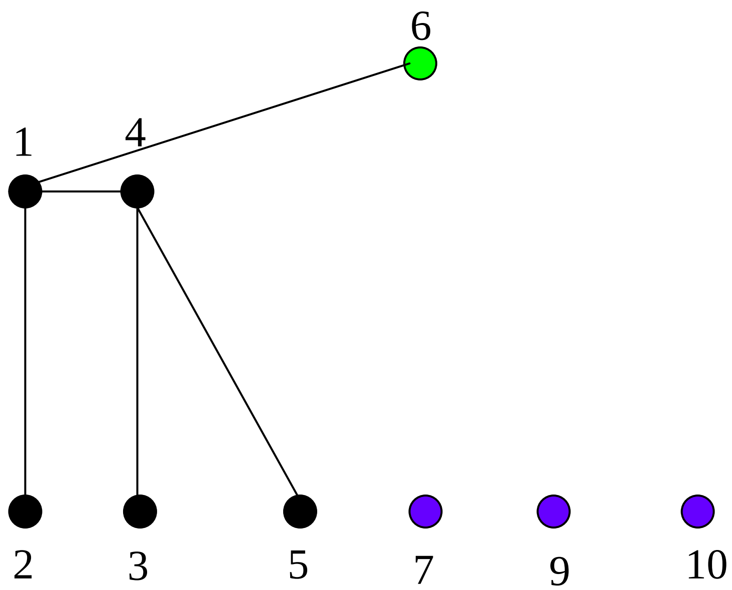

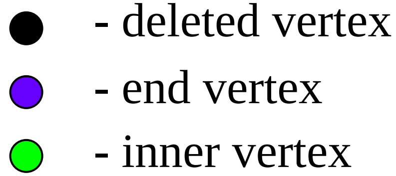

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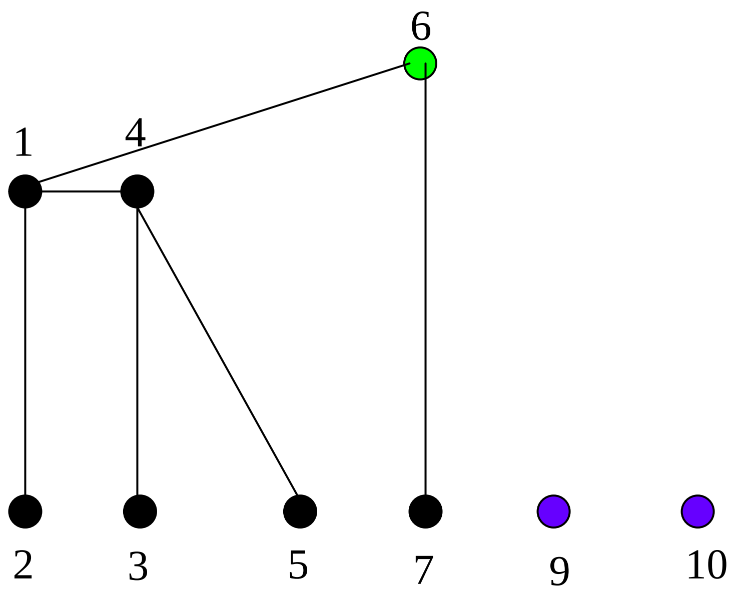

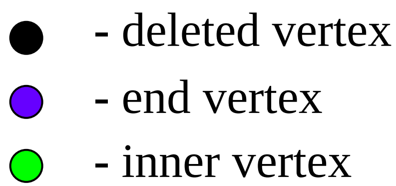

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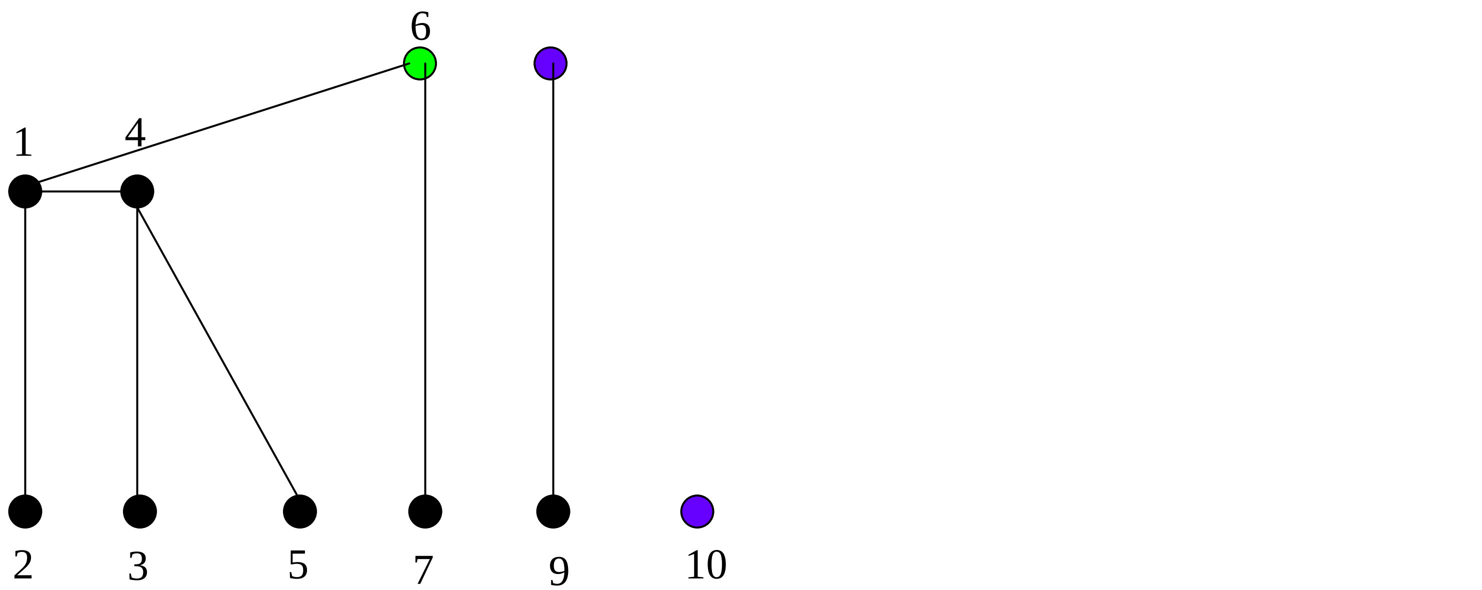

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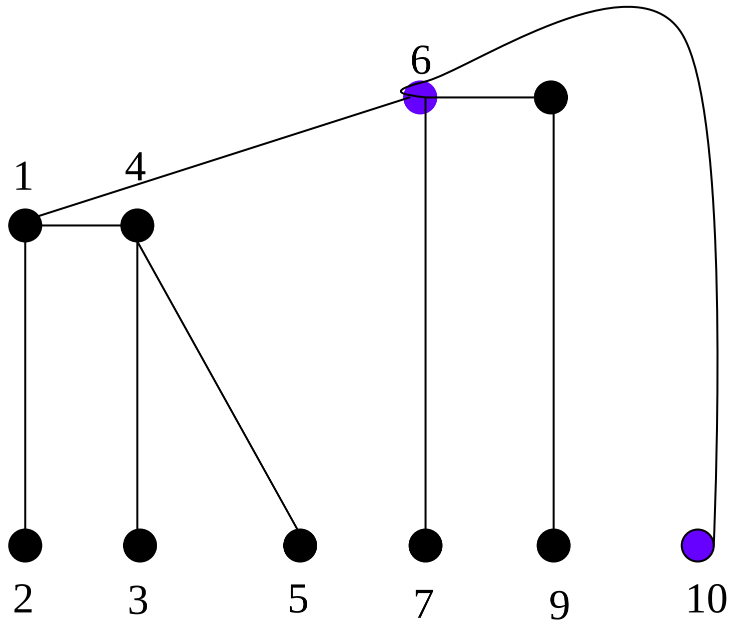
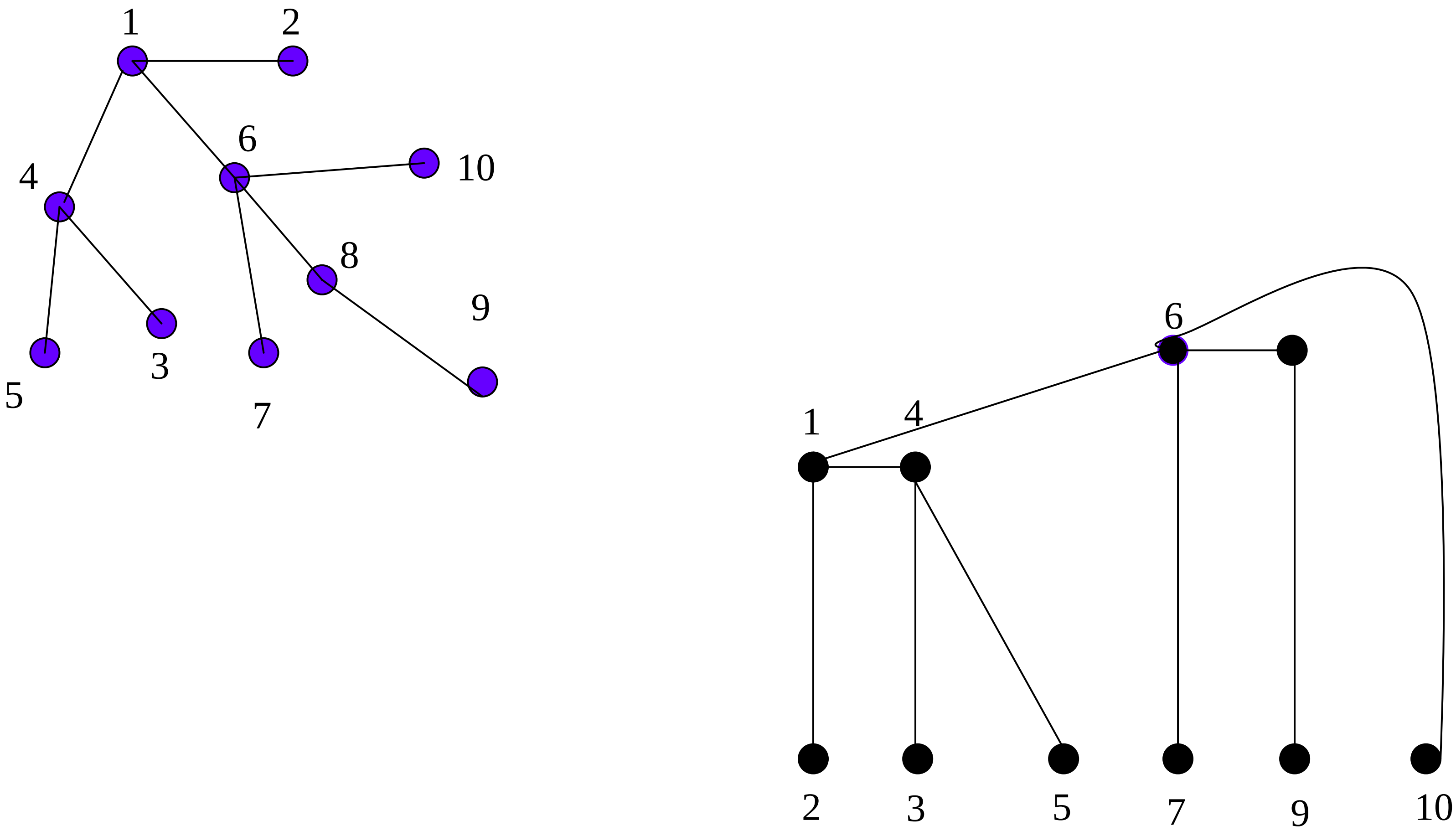


- deleted vertex
- end vertex
- inner vertex

Reversing the correspondence



Other applications: the number of trees with a given degree sequence

$$(x_1 + \ldots + x_m)^n = \sum_{\substack{(d_1,\ldots,d_m) \\ \sum_i d_i = n}} \frac{n!}{d_1! \cdot \ldots \cdot d_m!} x_1^{d_1} \cdot \ldots \cdot x_m^{d_m}$$

Let $(d_1,\ldots,d_n)$ be the degree sequence

$$\frac{(n-2)!}{(d_1 - 1)! \cdot \ldots \cdot (d_m - 1)!}$$

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$$\binom{n-2}{k-1}(n-1)^{n-k-1}$$ -the number of trees in which vertex n has degree k

Polya's approach

$$T(x) = \sum_{n=1}^{\infty} n t_n \frac{x^n}{n!}$$

T(x) is the generating function for the number of rooted trees with n vertices

Let c_n be the number of connected graphs on n vertices enjoying a certain property P.

$$\frac{1}{2}\sum_{k=1}^{n-1}\binom{n}{k}\cdot c_k \cdot c_{n-k} = \frac{1}{2}\cdot n!\sum_{k=1}^{n-1}\frac{c_k}{k!}\cdot\frac{c_{n-k}}{(n-k)!}$$

$$C(x) = \sum_{n=1}^{\infty} c_n \frac{x^n}{n!}$$

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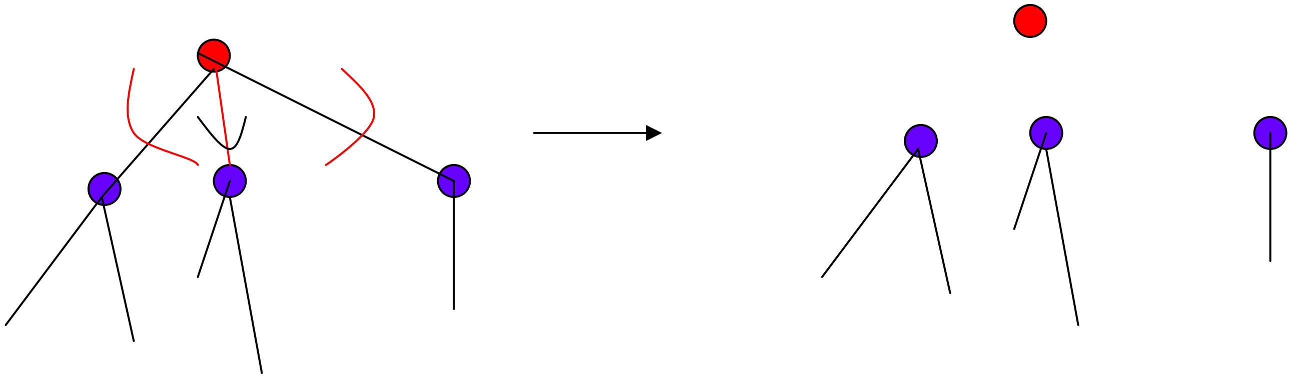


$$T(x) = xe^{T(x)}$$

Lagrange inversion formula

$$\varphi(s) = x\psi(\varphi(s))$$

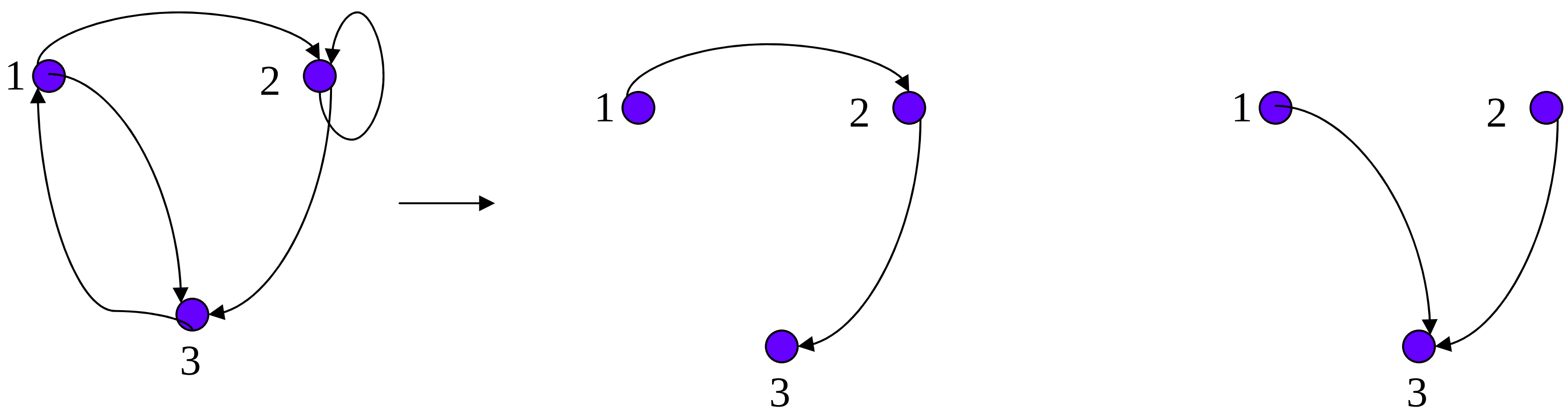
$$\frac{d^n}{ds^n}\varphi(s)\,|_{s=0} = \frac{1}{n}\frac{d^n}{dt^n}\psi^n(t)\,|_{t=0}$$

$$nt_n = \frac{1}{n}\frac{d^n}{dt^n}e^{nt}\,|_{t=0} = n^{n-1}$$

The number of spanning trees of a directed graph

Def. A spanning tree of a graph G is its subgraph T that includes all the vertices of G and is a tree

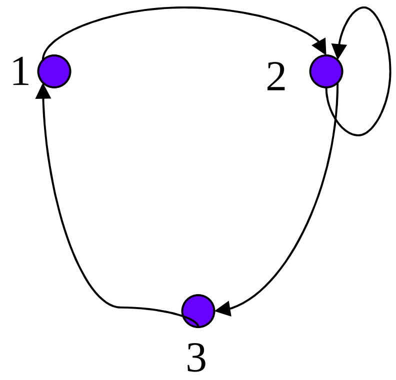
Def. A directed tree rooted at vertex n is a tree, all arcs of which are directed towards the root



Knuth's theorem

$$n = \sum_{j=1}^{h} s_j$$

Consider an example: $s_1 = 3, s_2 = 2, s_3 = 2$



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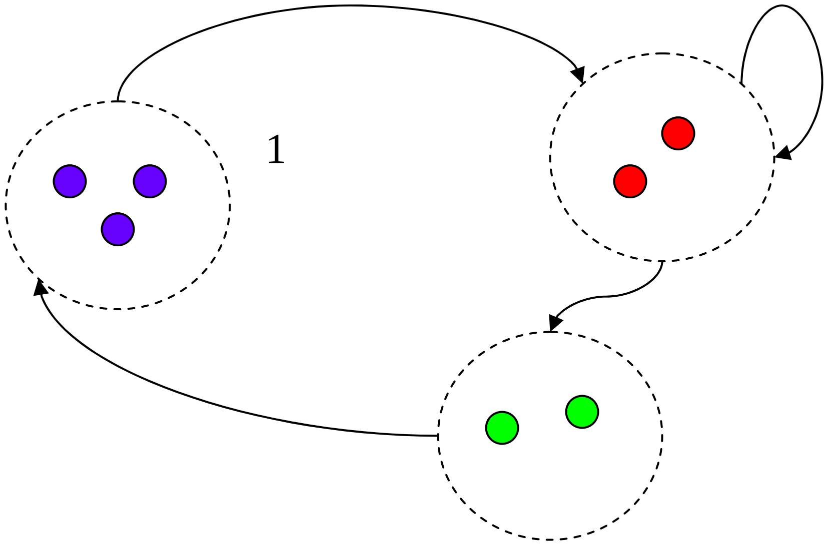

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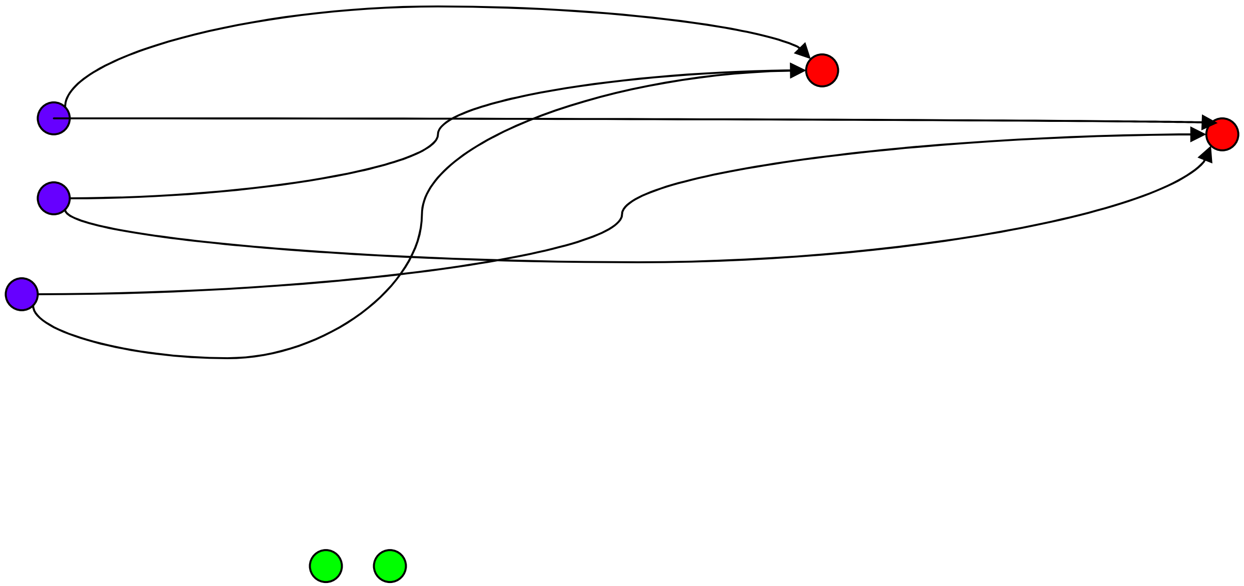

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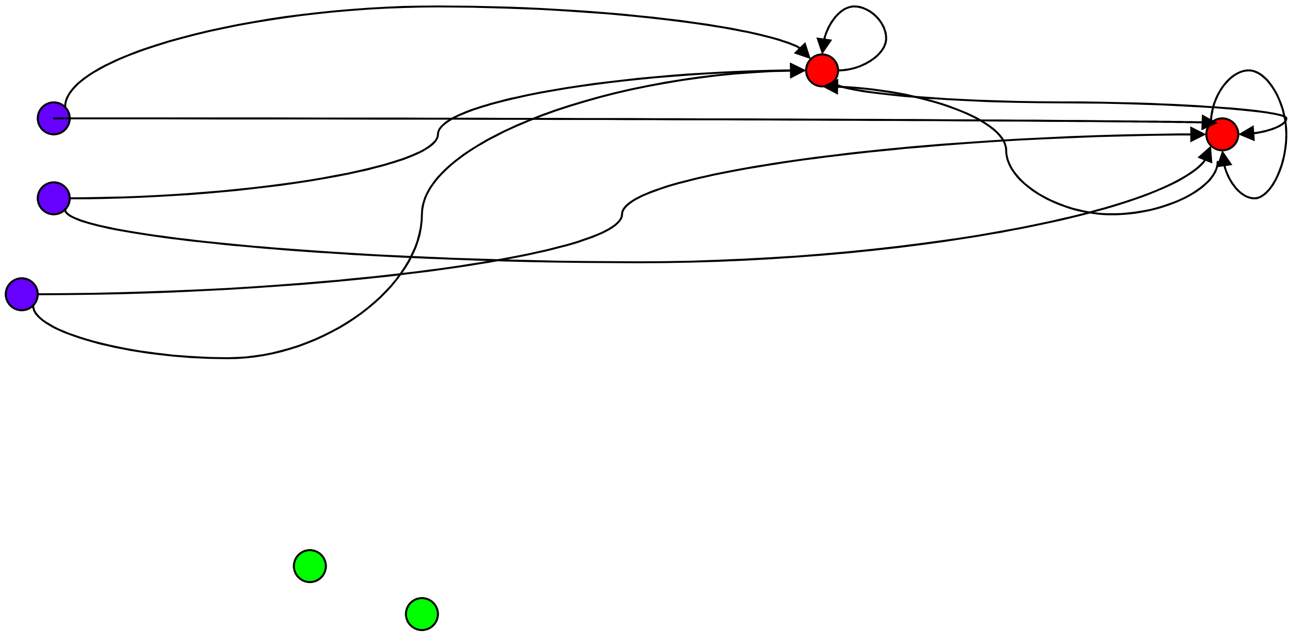

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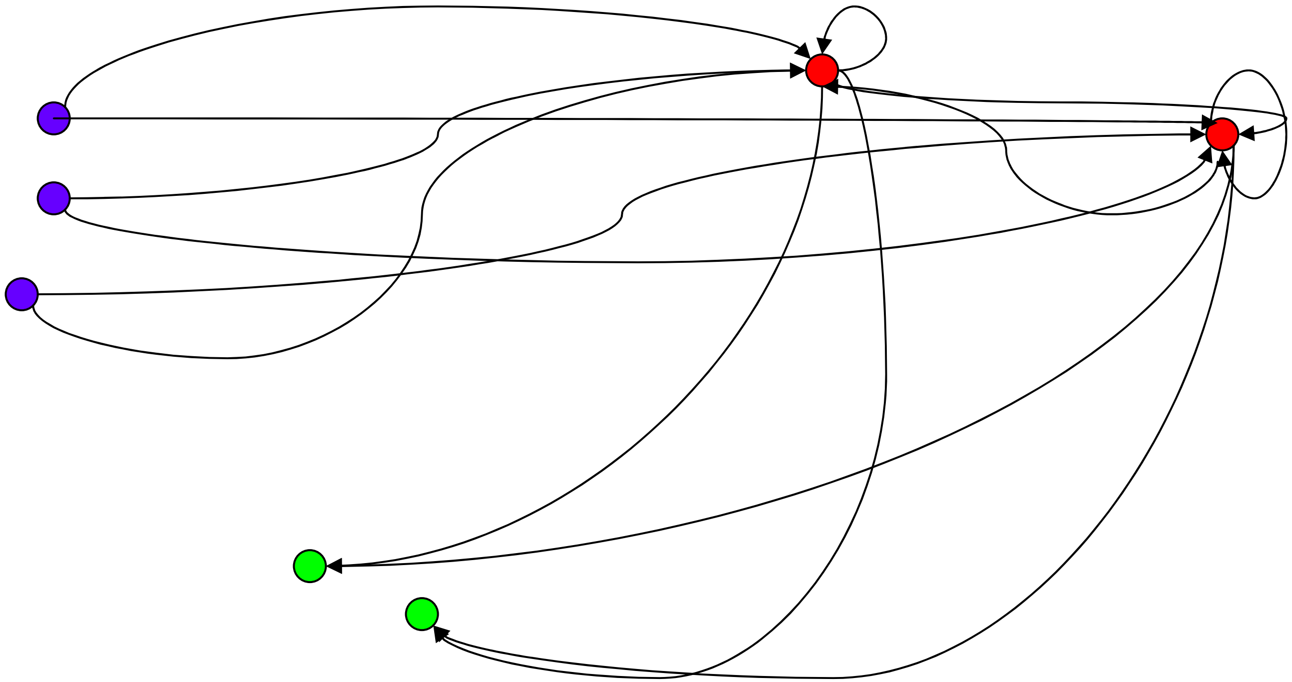

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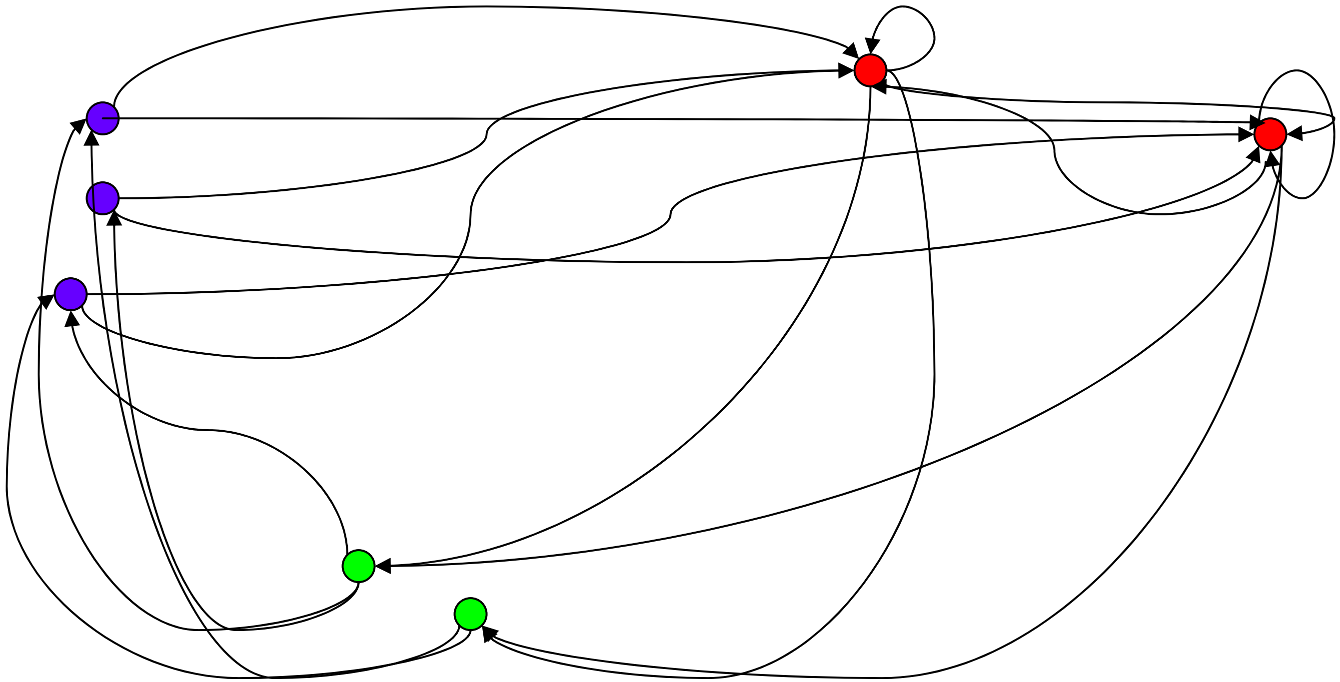
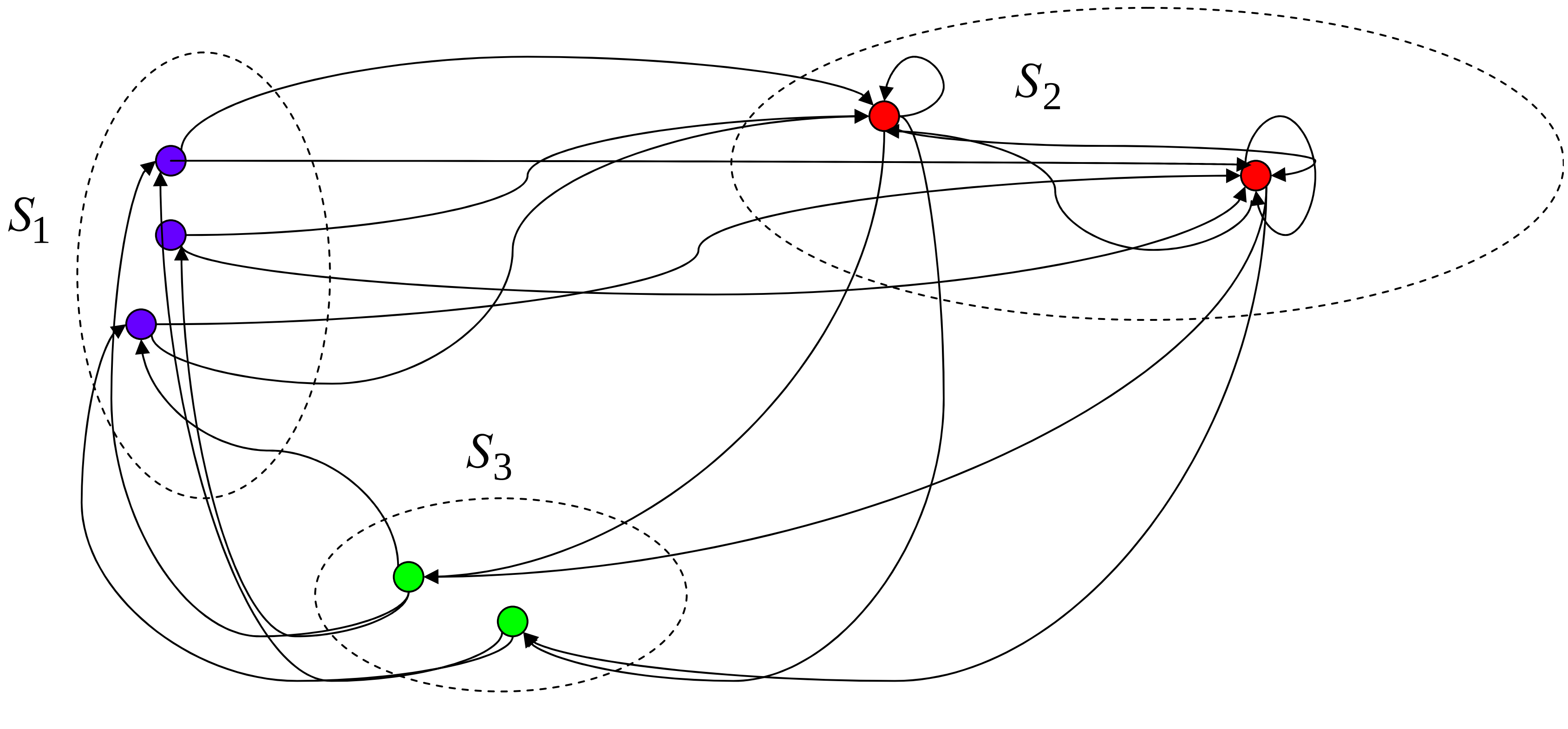


Knuth’s theorem

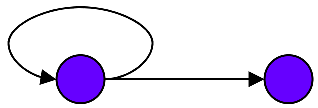
Def. A function f is called a tree function of a directed tree T iff f(i)=j when j is the first vertex on the way from i to the root.

Let c(H) denote the number of spanning trees of the graph H



Knuth's theorem

Theorem (Knuth) $c(H) = \sum_{f} \prod_{i=1}^{h-1} |\Gamma(S_i)|^{|S_i|-1} |f(S_i)|$



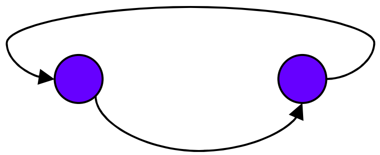
Theorem The number of spanning trees of a graph H arisen from a directed cycle equals

$$s_2^{s_1 - 1} \cdot s_3^{s_2} \cdot s_4^{s_3} \cdot \ldots \cdot s_1^{s_h - 1}$$

Knuth's theorem

<u>Theorem</u> The number of spanning trees of a graph H arisen from a directed cycle equals

$$s_2^{s_1-1} \cdot s_3^{s_2} \cdot s_4^{s_3} \cdot \ldots \cdot s_1^{s_h-1}$$



$s_2^{s_1-1} \cdot s_1^{s_2-1}$ is the number of r by s bipartite graphs

Knuth's theorem

$$\sum_{k=0}^{n} \binom{n}{k} k^{n-k-1}(n-k-1)^{k-1} = 2n^{n-2}$$

Matrix-Tree Theorem

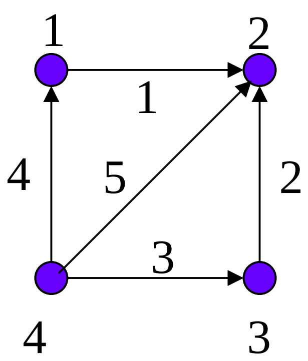
Def. Let G be a directed graph without loops. Let $\{v_1,\ldots,v_n\}$ denote the vertices of G, and $\{e_1,\ldots,e_m\}$ denote the edges of G.

The incidence matrix of G is the n x m matrix A, such that

$a_{i,j} = 1,$ if v_i is the head of e_j

$a_{i,j} = -1,$ if v_i is the tail of e_j

$a_{i,j} = 0$ otherwise

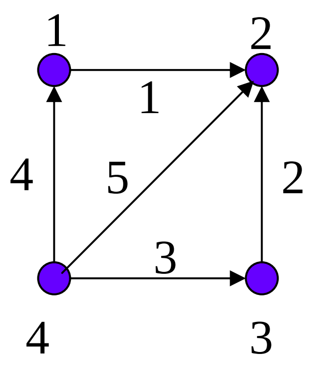


$+1$	0	0	-1	0
-1	-1	0	0	-1
0	$+1$	-1	0	0
0	0	$+1$	$+1$	$+1$

Matrix-Tree Theorem

<u>Lemma.</u> The incidence matrix of a connected graph on n vertices has the rank of n-1

The *reduced incidence matrix* A of a connected graph G is the matrix obtained from the incidence matrix by deleting a certain row.



+1	0	0	−1	0
−1	−1	0	0	−1
0	+1	−1	0	0
0	0	+1	+1	+1

Matrix-Tree Theorem

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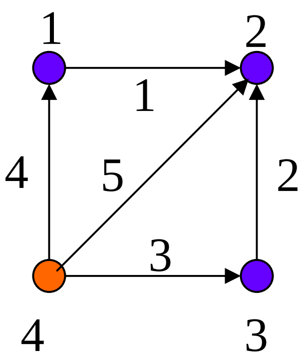

$$\begin{matrix} +1 & 0 & 0 & -1 & 0 \\ -1 & -1 & 0 & 0 & -1 \\ 0 & +1 & -1 & 0 & 0 \end{matrix}$$

Matrix-Tree Theorem

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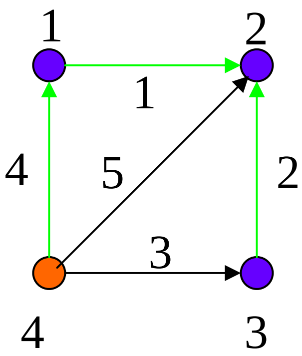

$$\begin{matrix} +1 & 0 & -1 \\ -1 & -1 & 0 \\ 0 & +1 & 0 \end{matrix}$$

Matrix-Tree Theorem

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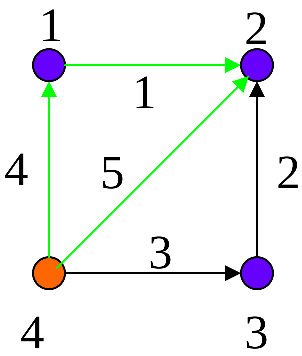


+1	−1	0
−1	0	−1
0	0	0

Matrix-Tree Theorem

Let A_0 be the reduced incidence matrix of the graph G.

Theorem (Binet-Cauchy)

If R and S are matrices of size p by q and q by p, where $p \leq q$, then

$$\det(RS) = \sum \det(B) \cdot \det(C)$$

Theorem (Matrix-Tree Theorem)

If A is a reduced incidence matrix of the graph G, then the number of spanning trees equals $\det(A \cdot A^T)$

$$\det A \;\; A^T = \sum (\det B)^2$$

Matrix-Tree Theorem

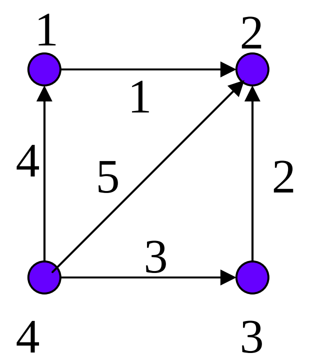
$e_1, \ldots e_b$ – variables identified with edges of G

$M(e) = [m_{ij}]$

$m_{ij} = -e_k$, if e_k joins i and j and $i \neq j$

$m_{ij} =$ sum if edges incident to i otherwise

<u>Theorem</u> $M_n(e) = \sum \Pi(T)$



$$M_4(e) = \begin{vmatrix} e_1 + e_4 & -e_1 & 0 \\ -e_1 & e_1 + e_2 + e_5 & -e_2 \\ 0 & -e_2 & e_2 + e_3 \end{vmatrix} =$$

$$= e_1e_2e_3 + e_1e_2e_4 + e_1e_2e_5 + e_1e_3e_4 + e_1e_3e_5 + e_2e_3e_4 + e_2e_4e_5 + e_3e_4e_5$$

Matrix-Tree Theorem

$e_1,...e_b - \text{var}\,iables \quad identified \quad with \quad edges \quad of \quad G$

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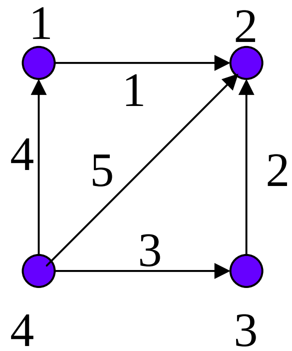
$m_{ij} = -e_k, if \quad e_k \quad joins \quad i \quad and \quad j \quad and \quad i \neq j$

$m_{ij} = sum \quad if \quad edges \quad incident \quad to \quad i \quad otherwise$

<u>Theorem</u> $M_n(e) = \sum \Pi(T)$

$M_n(e) = \det(A \cdot Y \cdot A^T)$



$$M_4(e) = \begin{vmatrix} e_1 + e_4 & -e_1 & 0 \\ -e_1 & e_1 + e_2 + e_5 & -e_2 \\ 0 & -e_2 & e_2 + e_3 \end{vmatrix} =$$

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Matrix-Tree Theorem

Another derivation of Cayley's formula:

$$\begin{vmatrix} n-1 & -1 & \dots & -1 & -1 \\ -1 & n-1 & \dots & -1 & -1 \\ -1 & -1 & \ddots & -1 & -1 \\ -1 & -1 & -1 & n-1 & -1 \\ -1 & -1 & -1 & -1 & n-1 \end{vmatrix}$$

Matrix-Tree Theorem

Another derivation of Cayley's formula:

$$\begin{vmatrix} 1 & 1 & \ldots & 1 & 1 \\ -1 & n-1 & \ldots & -1 & -1 \\ -1 & -1 & \ddots & -1 & -1 \\ -1 & -1 & -1 & n-1 & -1 \\ -1 & -1 & -1 & -1 & n-1 \end{vmatrix}$$

Matrix-Tree Theorem

Another derivation of Cayley's formula:

$$\begin{vmatrix} 1 & 1 & \dots & 1 & 1 \\ 0 & n & \dots & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & n & 0 \\ 0 & 0 & 0 & 0 & n \end{vmatrix} = n^{n-2}$$

Matrix-Tree Theorem

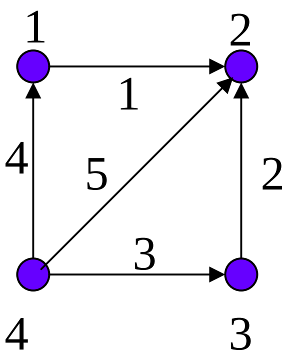
Theorem (Matrix-Tree Theorem for directed graphs)

Let be variables representing the arcs of the graph. Let $C = [c_{ij}]$ denote the n by n matrix in which $-c_{ij}$ equals the sum of arcs directed from node i to node j if $i \neq j$, and c_{ii} equals the sum of all arcs directed from node i to all other nodes.

Then

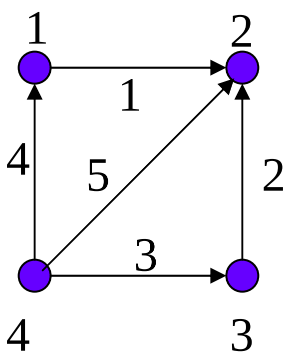
$$C_n = \sum \Pi(T),$$

where the summation is over all spanning subtrees of G rooted at node n .



$$C_n = \begin{pmatrix} e_1 & -e_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -e_2 & e_2 & 0 \\ -e_4 & -e_5 & -e_3 & e_3 + e_4 + e_5 \end{pmatrix}$$

Matrix-Tree Theorem



$$C_2 = \begin{vmatrix} e_1 & 0 & 0 \\ 0 & e_2 & 0 \\ -e_4 & -e_3 & e_3 + e_4 + e_5 \end{vmatrix} = e_1 e_2 e_3 + e_1 e_2 e_4 + e_1 e_2 e_5$$