Window Fourier and Wavelet Transforms. Properties and Applications of Wavelets

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1 Introduction

Nowadays, wavelets are useful and quite modern tool of applied mathematics which has many applications especially in data processing and compression. The simplicity of wavelets makes them almost perfect for some special purposes. It is well known that conventional Fourier Transform and the Window Fourier Transform (WFT) are of extensive use for data processing and compression. The motivation of using wavelets for data processing is a possibility to have a flexible resolution depending on the details of the data time evolution. This feature referred to as Multi Resolution Analysis is a main advantage of wavelet approach in comparison to WFT since the latter does not allow different levels of resolution for different time and frequencies regions. Other advantageous features of wavelets are orthonormality, compactness of the basis functions support (in contrast to sinuses and cosines). The aim of this work is to present a brief introduction to WFT analysis and wavelet analysis and to compare these methods.

2 Window Fourier Transform

In this section we introduce the Window Fourier Transform (WFR). Let $f(t)$ be the absolute integrable function on $\mathbf{R}$ then the ordinary Fourier Transform (FT) is defined [3] as the following integral

$$F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt \tag{1}$$

and the inverse transform is given by

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\omega) e^{i\omega t} d\omega. \tag{2}$$

In context of data processing the function $f(t)$ is commonly referred to as a time signal whereas the $F(\omega)$ as a frequency spectrum. The FT allows us to obtain the Fourier spectrum $F(\omega)$ of the signal $f(t)$. This spectrum $F(\omega)$ is the global characteristic of the signal and contribution of local properties of $f(t)$ in $F(\omega)$ is of very "integral" nature. This means that it is very difficult (or impossible) to find explicitly which part of the time region and what properties of the signal $f(t)$ in this region are responsible for the local behavior of the spectrum $F(\omega)$. This can be partially improved by the Window Fourier Transform (WFT) which has the form

$$T^{win} f(\omega, s) = \int_{-\infty}^{\infty} f(t) g(t-s) e^{-i\omega t} dt. \tag{3}$$

Here $g(t)$ is the so called window function which allows to see how spectrum changes through the time.

The integral (3) is often very complicated to evaluate for all values of parameters ω and s. It is why, the discrete form of (3)

$$T^{win}_{m,n} f(\omega_0, s_0) = \int_{-\infty}^{\infty} f(t) g(t - ns_0) e^{-im\omega_0 t} dt, \qquad m, n \in \mathbf{Z} \tag{4}$$

is useful for certain applications. This formula defines the WFT for values of ω and s belonging to the equidistant two dimensional grid $\{m\omega_0,\ ns_0,\ m,n \in \mathbf{Z}\}$. The area of each sell of the grid depends only on the window function $g(t)$ and does not on the resolution. The analysis which is out of the scope of this work, shows that the parameters of the sell should obey the inequality $s_0\omega_0 \geq 1/4\pi$, which can be interpreted as a Heisenberg uncertainty principle.

For applications, the constant area of a cell of the grid is not a restrictive factor whereas the constant spacings ω_0, s_0 make the WFT not flexible enough. For example, for low frequencies the "wide" window is more appropriate because the signal changes slowly, and for high frequencies a "thin" is more adequate. This flexibility can be achieved by using the formalism of Multi Resolution Analysis which we describe in the next section.

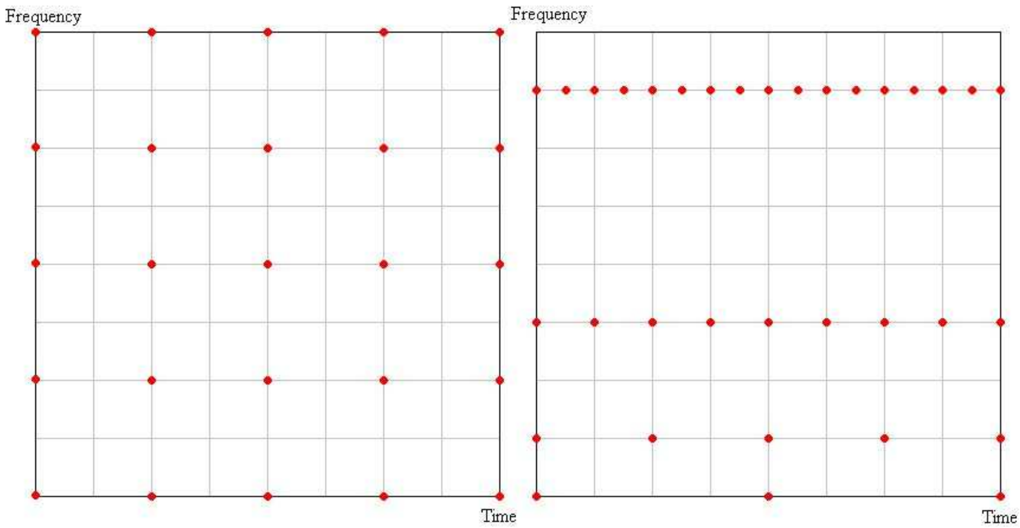

FIG. 1: Disadvantage of Window Fourier Method

3 Multi Resolution Analysis

Multi Resolution Analysis [1],[5] is a sequence of closed subspaces $\{V_j\}$ (they are called approximation subspaces) of special kind with following properties

1. $V_j \subset V_{j+1}$
2. $\bigcup_{j\in Z} V_j = L^2(\mathbf{R})$
3. $\bigcap_{j\in Z} V_j = \{0\}$
4. if $f(t) \in V_j \Rightarrow f(2t) \in V_{j+1}$
5. if $f(t) \in V_j \Rightarrow f(t-k) \in V_j$
6. single father function φ defines orthonormal basis in corresponding subspace V_j by scaling and translations

$$\varphi_{j,k}(t) = 2^{j/2}\varphi(2^j t - k). \tag{5}$$

Equation (5) is called *scaling equation.* It is easy to see, that basis functions of V_j subspace can be represented in terms of basis functions of more "fine" subspace V_{j+1} as following

$$\varphi_{j,k}(t) = \frac{1}{\sqrt{2}} \sum_{k\in\mathbf{Z}} h_k \varphi_{j+1,k}(t). \tag{6}$$

Let us define now new subspaces W_i which are linear complement of V_i in V_{i+1}, *i.e.*,

$$V_i + W_i = V_{i+1}. \tag{7}$$

Obviously, the basis of W_i formally can be constructed by a formula similar to (6)

$$\psi_{i,k}(t) = \frac{1}{\sqrt{2}} \sum_{k \in \mathbf{Z}} g_k \varphi_{i+1,k}(t). \tag{8}$$

Basis functions $\psi_{i,k}(t)$ are called *wavelets*. The expansion coefficients h_k in (6) as well as g_k in (8) are the respective projections and related to each other by the formula

$$g_i = (-1)^i h_{L-i-1} \qquad i = 1 \dots L-1.$$

These coefficients are called as *filter coefficients* with filter length L. The above constructions can be illustrated by the following picture

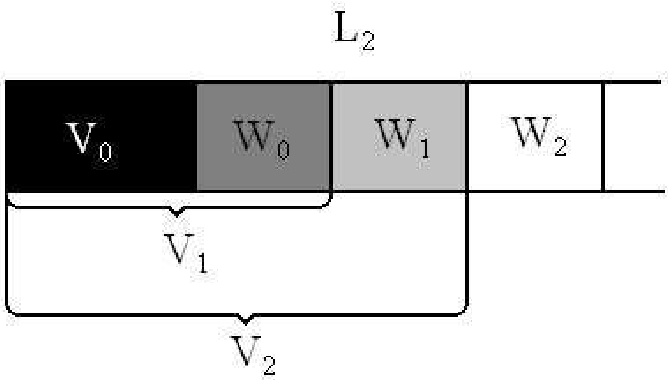

FIG. 2: Subspaces V and W

3.1 Fast Wavelet transform

Similarly to FT, the strategy of fast transform can be implemented in practical calculations with wavelets. This strategy is described in this section and in the following section for respective inverse transform. The general form of wavelet transform for a function $f(t)$ can be written as follows

$$f(t) = \sum_{j=L}^{J-1} \sum_{k=0}^{2^j-1} w_{j,k} \psi_{J,k}(t) + \sum_{k=0}^{L-1} s_{J,k} \varphi_{J,k}(t). \tag{9}$$

Due to orthonormality of the wavelets basis the expansion coefficients are given by projections

$$\begin{aligned} s_{j,k} &= \textstyle\int_{-\infty}^{\infty} f(t) \overline{\varphi}_{j,k}(t) dt, \\ w_{j,k} &= \textstyle\int_{-\infty}^{\infty} f(t) \overline{\psi}_{j,k}(t) dt. \end{aligned} \tag{10}$$

To arrive at expansion (9) we will start from the following representation

$$f(t) = \sum_{k=0}^{2^J-1} s_{J,k}\varphi_{0,k}(t) \tag{11}$$

As $s_{0,k}$ coefficients we can use values of function $f(t)$ on equidistant grid. After that, we can convert the expansion from V_J to $V_{J-1}+W_{J-1}$. The $w_{J-1,k}$ coefficients by definition are the integrals

$$w_{J-1,k} = \int_{-\infty}^{\infty} f(t)\overline{\psi}_{J-1,k}(t)dt.$$

Then, using (8) we can get

$$\int_{-\infty}^{\infty} f(t)\overline{\psi}_{J-1,k}(t)dt = \int_{-\infty}^{\infty} f(t)\sum_{l\in\mathbf{Z}} \overline{g}_{l-2k}\overline{\varphi}_{J,l}(t)dt.$$

Now, interchanging summation and integration we obtain

$$\int_{-\infty}^{\infty} f(t)\sum_{l\in\mathbf{Z}} \overline{g}_{l-2k}\overline{\varphi}_{J,l}(t)dt = \sum_{l\in\mathbf{Z}} \overline{g}_{l-2k}\int_{-\infty}^{\infty} f(t)\overline{\varphi}_{J,l}(t)dt.$$

Finally, we arrive at the formula

$$\sum_{l\in\mathbf{Z}} \overline{g}_{l-2k}\int_{-\infty}^{\infty} f(t)\overline{\varphi}_{J,l}(t)dt = \sum_{l\in\mathbf{Z}} \overline{g}_{l-2k}s_{J,l}.$$

By similar way we can get the following formulae

$$s_{J-1,k} = \int_{-\infty}^{\infty} f(t)\overline{\varphi}_{J-1,k}(t)dt,$$

$$\int_{-\infty}^{\infty} f(t)\overline{\varphi}_{J-1,k}(t)dt = \int_{-\infty}^{\infty} f(t)\sum_{l\in\mathbf{Z}} \overline{h}_{l-2k}\overline{\varphi}_{J,l}(t)dt,$$

$$\int_{-\infty}^{\infty} f(t)\sum_{l\in\mathbf{Z}} \overline{h}_{l-2k}\overline{\varphi}_{J,l}(t)dt = \sum_{l\in\mathbf{Z}} \overline{h}_{l-2k}\int_{-\infty}^{\infty} f(t)\overline{\varphi}_{J,l}(t)dt,$$

$$\sum_{l\in\mathbf{Z}} \overline{h}_{l-2k}\int_{-\infty}^{\infty} f(t)\overline{\varphi}_{J,l}(t)dt = \sum_{l\in\mathbf{Z}} \overline{h}_{l-2k}s_{J,l}.$$

So that, we have obtained the following relations between expansion coefficients

$$w_{j-1,k} = \sum_{l\in\mathbf{Z}} \overline{g}_{l-2k}s_{j,l}$$

$$s_{j-1,k} = \sum_{l \in \mathbf{Z}} \overline{h}_{l-2k} s_{j,l}$$

which form the basis for fast wavelet transform.

As illustration, suppose that filter length is finite (let say equal to 4) and cyclic boundary conditions are used, then the matrix which corresponds to the (9) has the structure

$$T = \begin{pmatrix} h_1 & h_2 & h_3 & h_4 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & h_1 & h_2 & h_3 & h_4 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & h_1 & h_2 & h_3 & h_4 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & h_1 & h_2 & h_3 & h_4 \\ h_3 & h_4 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & h_1 & h_2 \\ g_1 & g_2 & g_3 & g_4 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & g_1 & g_2 & g_3 & g_4 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & g_1 & g_2 & g_3 & g_4 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & g_1 & g_2 & g_3 & g_4 \\ g_3 & g_4 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & g_1 & g_2 \end{pmatrix}$$

and its action realizing the wavelet transform can be illustrated by the following chain of equalities

$$T \begin{pmatrix} s_{i+1,0} \\ s_{i+1,1} \\ s_{i+1,2} \\ s_{i+1,3} \\ s_{i+1,4} \\ s_{i+1,5} \\ s_{i+1,6} \\ s_{i+1,7} \end{pmatrix} = \begin{pmatrix} s_{i,0} \\ s_{i,1} \\ s_{i,2} \\ s_{i,3} \\ w_{i,0} \\ w_{i,1} \\ w_{i,2} \\ w_{i,3} \end{pmatrix} = \begin{pmatrix} s_{i,0} \\ s_{i,1} \\ s_{i,2} \\ s_{i,3} \end{pmatrix} + \begin{pmatrix} w_{i,0} \\ w_{i,1} \\ w_{i,2} \\ w_{i,3} \end{pmatrix}$$

and with the following figure

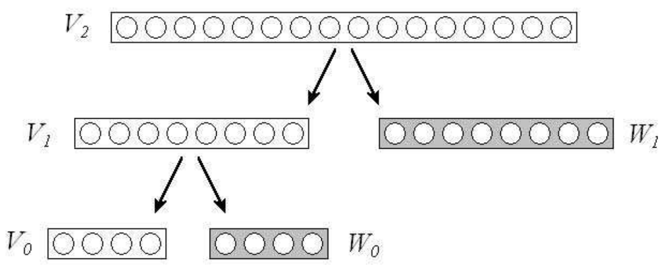

FIG. 3: Fast Wavelet Transform

3.2 Inverse Fast Wavelet Transform

The fast inverse wavelet transform is based on the following representations. Let us note that since

$$V_{j+1} = V_j \bigoplus W_j$$

we can get the representation for $\varphi_{j+1,k}$ in the form

$$\varphi_{j+1,k} = \sum_{l\in\mathbf{Z}} a_l \varphi_{j,l} + \sum_{l\in\mathbf{Z}} b_l \psi_{j,l} \tag{12}$$

where

$$a_l = \int_{-\infty}^{\infty} \varphi_{j+1,k}(t)\varphi_{j,k}(t)dt.$$

Now using (6) and (8) we can rewrite the right hand side in the following way

$$a_l = \int_{-\infty}^{\infty} \varphi_{j+1,k}(t) \sum_{m\in\mathbf{Z}} h_{m-2l}\varphi_{j+1,k}(t)dt.$$

Interchanging integration and summation and taking into account the equality

$$\sum_{m\in\mathbf{Z}} h_{m-2l} \int_{-\infty}^{\infty} \varphi_{j+1,k}(t)\varphi_{j+1,k}(t)dt = h_{k-2l}$$

we get for a_l

$$a_l = h_{k-2l}.$$

In the similar way we get

$$b_l = g_{k-2l}.$$

These two representations allow as to rewrite the formula (12) in the form

$$\varphi_{j+1,k} = \sum_{l\in\mathbf{Z}} h_{k-2l}\varphi_{j,l} + \sum_{l\in\mathbf{Z}} g_{k-2l}\psi_{j,l}.$$

The matrix corresponding to the inverse transform in the conditions of preceding subsection has the form

$$
T^{inv} = \begin{pmatrix}
h_1 & 0 & 0 & \cdots & 0 & h_3 & g_1 & 0 & 0 & \cdots & 0 & g_3 \\
h_2 & 0 & 0 & \cdots & 0 & h_4 & g_2 & 0 & 0 & \cdots & 0 & g_4 \\
h_3 & h_1 & 0 & \cdots & 0 & 0 & g_3 & g_1 & 0 & \cdots & 0 & 0 \\
h_4 & h_2 & 0 & \cdots & 0 & 0 & g_4 & g_2 & 0 & \cdots & 0 & 0 \\
0 & h_3 & h_1 & \cdots & 0 & 0 & 0 & g_3 & g_1 & \cdots & 0 & 0 \\
0 & h_4 & h_2 & \cdots & 0 & 0 & 0 & g_4 & g_2 & \cdots & 0 & 0 \\
0 & 0 & h_3 & \cdots & 0 & 0 & 0 & 0 & g_3 & \cdots & 0 & 0 \\
0 & 0 & h_4 & \cdots & 0 & 0 & 0 & 0 & g_4 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & h_1 & 0 & 0 & 0 & 0 & \cdots & g_1 & 0 \\
0 & 0 & 0 & \cdots & h_2 & 0 & 0 & 0 & 0 & \cdots & g_2 & 0 \\
0 & 0 & 0 & \cdots & h_3 & h_1 & 0 & 0 & 0 & \cdots & g_3 & g_1 \\
0 & 0 & 0 & \cdots & h_4 & h_2 & 0 & 0 & 0 & \cdots & g_4 & g_2
\end{pmatrix}
$$

It is worth mentioning, that due to the orthonormality of the wavelets the corresponding matrices are orthogonal, *i.e.*,

$$T^{inv} = T^t = T^{-1}$$

which can be verified explicitly.

4 Conditions on wavelets

In order to get the correct Multi Resolution Analysis with required properties the father function and, consequently, wavelets have to obey the appropriate conditions. Ultimately, the primary condition is the scaling equation for wavelet. Let us list the possible set of additional conditions

1. *Compact Support.*
   Theorem: If wavelet has nonzero coefficients with only indexes from n to $n+m$ then the father function support is concentrated on the interval $[n, n+m]$.

2. *Orthogonality.*
   This means
   $$\int \varphi(x)\varphi(x+n)dx = \delta_{0,n}.$$
   Here $\delta_{0,n} = 1$ if $n = 0$ and zero otherwise. This orthogonality can be transformed into property for coefficients
   $$\sum_{k\in Z} h_k h_{k+2l} = 0 \quad l \neq 0.$$

3. *Zero momentums of father function and wavelet.*
   The momentums of Father function and wavelet are defined as integrals

$$M_i = \int x^i \varphi(x) dx,$$

$$\mu_i = \int x^i \psi(x) dx$$

   Zero momentums make function more smooth and differentiable. Note, since we have (8) then if $\varphi \in C^i$ it leads to $\psi \in C^i$. This condition can be rewritten in more simple form, for example

$$M_1 = \sum_{k \in Z} k h_k = 0$$

   and

$$\mu_1 = \sum_{k \in Z} k g_k = 0.$$

It is also useful to have fulfilled the following requirements

1. Symmetry of father function, if father function is symmetric then coefficients h_i will be also symmetric.

2. Rational coefficients.

5 Types of wavelets

In this section we will discuss the most popular types of wavelets, their properties and plots.

5.1 Haar wavelet (A. Haar)

Haar wavelet is the most simple wavelet. The only additional condition on this wavelet is *orthogonality*. So, we get only two conditions (one more is to satisfy scaling equation) and two equations

$$\begin{cases} h_0 + h_1 = \sqrt{2} \\ h_0^2 + h_1^2 = 1 \end{cases}.$$

The solution to these equations is

$$h_1 = h_2 = \frac{1}{\sqrt{2}}.$$

The subspaces V in this case are the spaces of piecewise constant functions.

Theorem: The only orthogonal basis with the symmetric, compactly supported father-function is the Haar basis.
Proof: Suppose $h = [\ldots]$. In general case the orthogonality is equivalent to the condition

$$\sum_{k \in Z} h_k h_{k+2l} = 0 \quad l \neq 0.$$

If $l = 2n$ then

$$a_n a_{n-1} + a_{n-1} a_n = 0,$$

and if $l = 2n - 2$ then

$$a_n a_{n-3} + a_{n-1} a_{n-2} + a_{n-2} a_{n-1} + a_{n-3} a_n = 0,$$

and so on. The only possible sequences are of the form

$$[\ldots 0, 0, \frac{1}{\sqrt{2}}, 0, 0 \ldots 0, 0, \frac{1}{\sqrt{2}}, 0, 0 \ldots].$$

Among these possibilities only the Haar filter leads to convergence in the solution of scaling (dilation) equation.

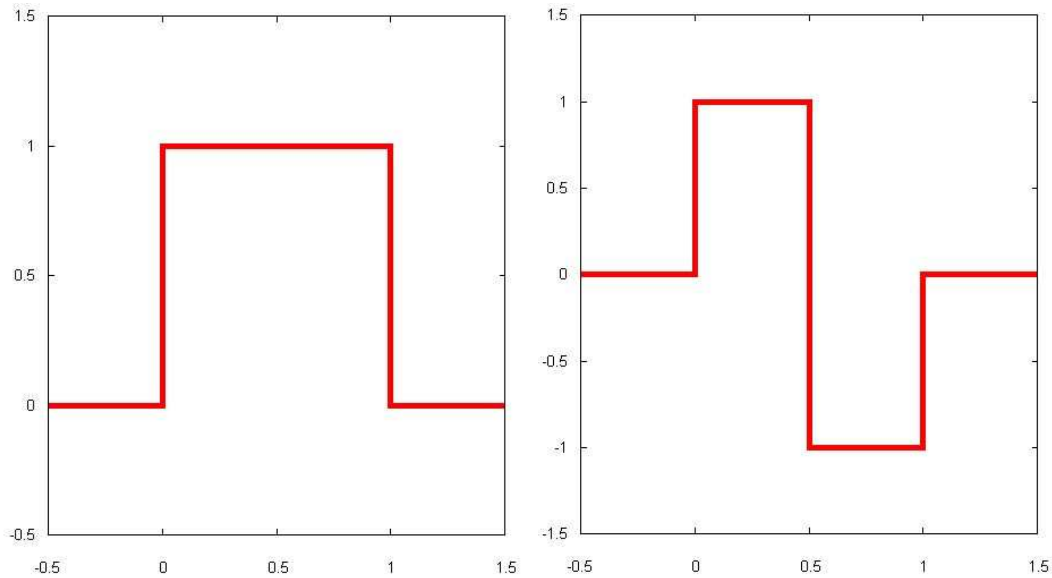

FIG. 4: Haar father and wavelet functions

5.2 Daubechies wavelets (I. Daubechies)

If we apply **zero moments** condition on father function we will get Daubechies family of wavelets. The Haar wavelet is the simplest Daubechies wavelet D2 (non zero moments). Now if we require the first moment to be zero we will get 4 conditions. Explicitly, these conditions are zero first momentum of father function; scaling equation; the orthogonality. These conditions for Daubechies D4 wavelet lead for the following set of equations

$$\begin{cases} h_0 + h_1 + h_2 + h_3 = \sqrt{2} \\ h_1 + 2h_2 + 3h_3 = 0 \\ h_0^2 + h_1^2 + h_2^2 + h_3^2 = 1 \\ h_0 h_2 + h_1 h_3 = 0 \end{cases}$$

The solution have the form

$$h_0 = \frac{1+\sqrt{3}}{4\sqrt{2}}, \quad h_1 = \frac{3+\sqrt{3}}{4\sqrt{2}}, \quad h_2 = \frac{3-\sqrt{3}}{4\sqrt{2}}, \quad h_3 = \frac{1-\sqrt{3}}{4\sqrt{2}}.$$

Here are the plots of the functions

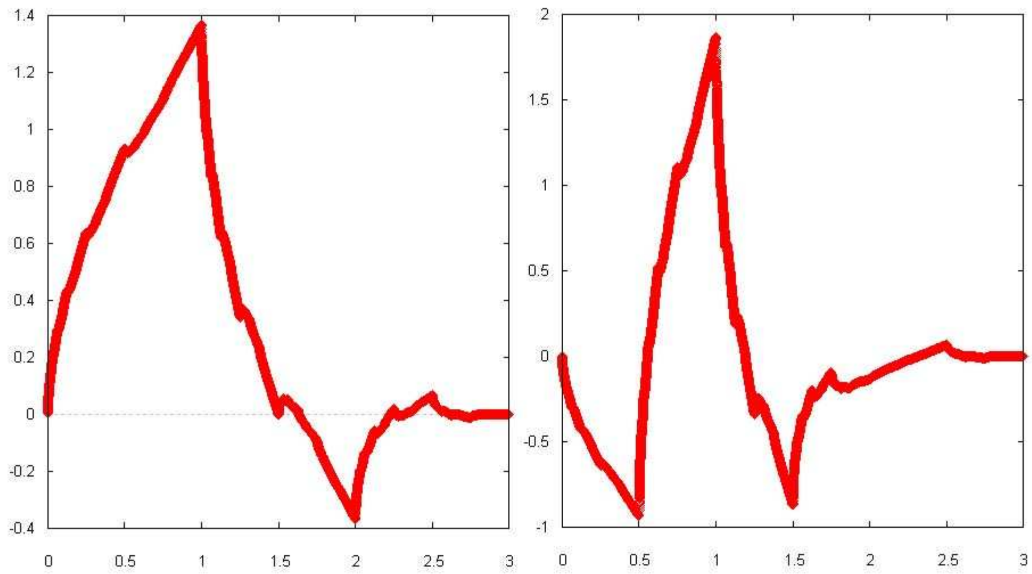

FIG. 5: Daubechies D4 father and wavelet function

If we require additionally the second momentum to be zero we will get Daubechies D6 wavelet with six coefficients enumerated from 0 to 5 (zero second momentum condition and additional orthogonality condition). Here is the respective plot

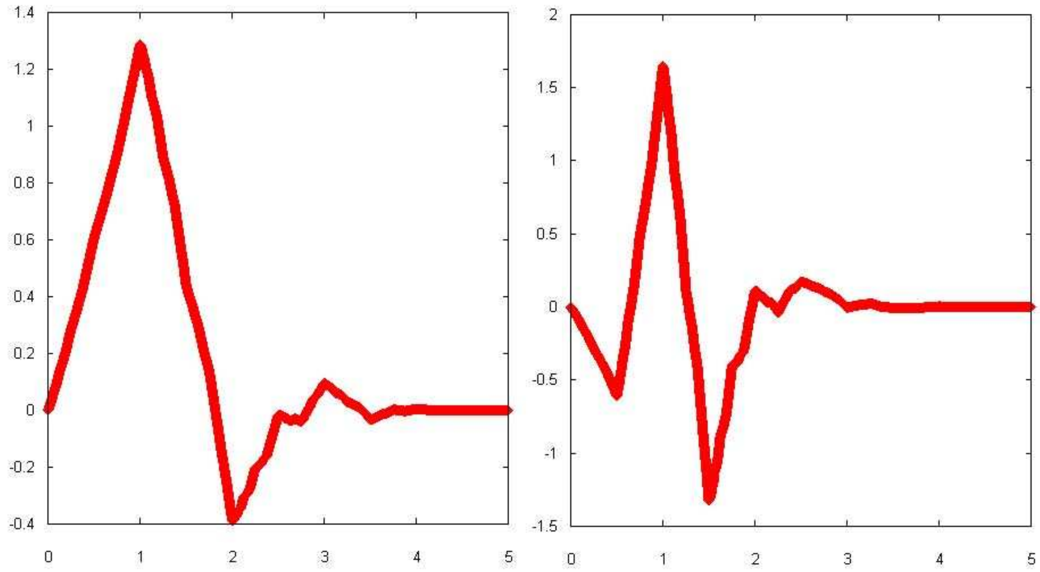

FIG. 6: Daubechies D6 father and wavelet function

5.3 Coiflets (R. Coifman)

If we consider pair of conditions, namely, zero momentum of father function and zero momentum of wavelet function we will get the family of Coiflets. The set of equations for coefficients for C2 Coiflet is

$$\begin{cases} h_{-2} + h_{-1} + h_0 + h_1 + h_2 + h_3 = \sqrt{2} \\ -2h_{-2} - h_{-1} + h_1 + 2h_2 + 3h_3 = 0 \\ -2h_{-2} + h_{-1} - h_1 + 2h_2 - 3h_3 = 0 \\ h_{-2}^2 + h_{-1}^2 + h_0^2 + h_1^2 + h_2^2 + h_3^2 = 1 \\ h_{-2}h_0 + h_{-1}h_1 + h_0h_2 + h_1h_3 = 0 \\ h_{-2}h_2 + h_{-1}h_3 = 0 \end{cases} .$$

The solution to this set is

$$h_{-2} = \frac{\sqrt{2} - \sqrt{14}}{32}, h_{-1} = \frac{-11\sqrt{2} + \sqrt{14}}{32}, h_0 = \frac{7\sqrt{2} + \sqrt{14}}{16},$$

$$h_1 = \frac{-\sqrt{2} - \sqrt{14}}{16}, h_2 = \frac{\sqrt{2} - \sqrt{14}}{32}, h_3 = \frac{-3\sqrt{2} + \sqrt{14}}{32}.$$

Here are plots of the functions

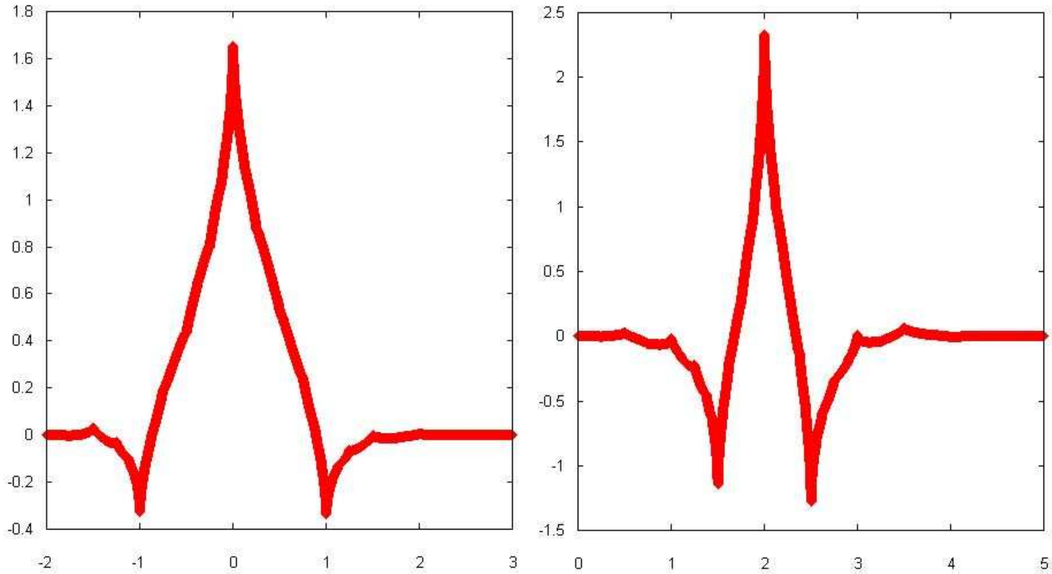

FIG. 7: Coiflet C2 father and wavelet function

If we add a pair of zero second momentums conditions then we will get C4 Coiflet with 12 coefficients enumerated from -4 to 7.

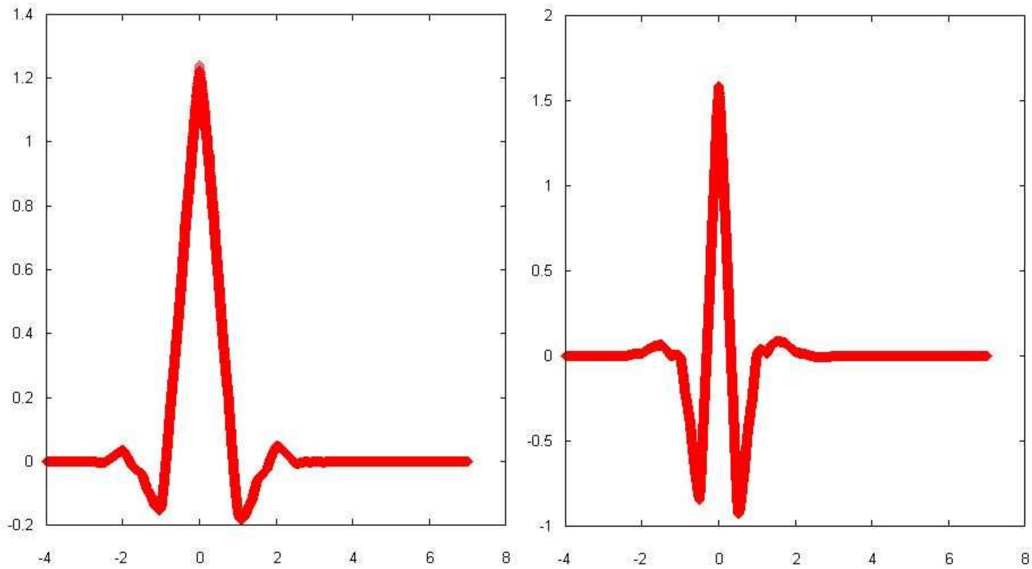

FIG. 8: Coiflet C4 scaling and wavelet function

5.4 Shannon wavelet (C. Shannon)

Shannon wavelet belongs to the small group of wavelet which has their father function represented in elementary functions

$$\varphi(x) = \mathrm{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$$

$$\psi(x) = 2\mathrm{sinc}(2x) - \mathrm{sinc}(x) = \frac{\sin(2\pi x) - \sin(\pi x)}{\pi x}$$

The graph of this function is

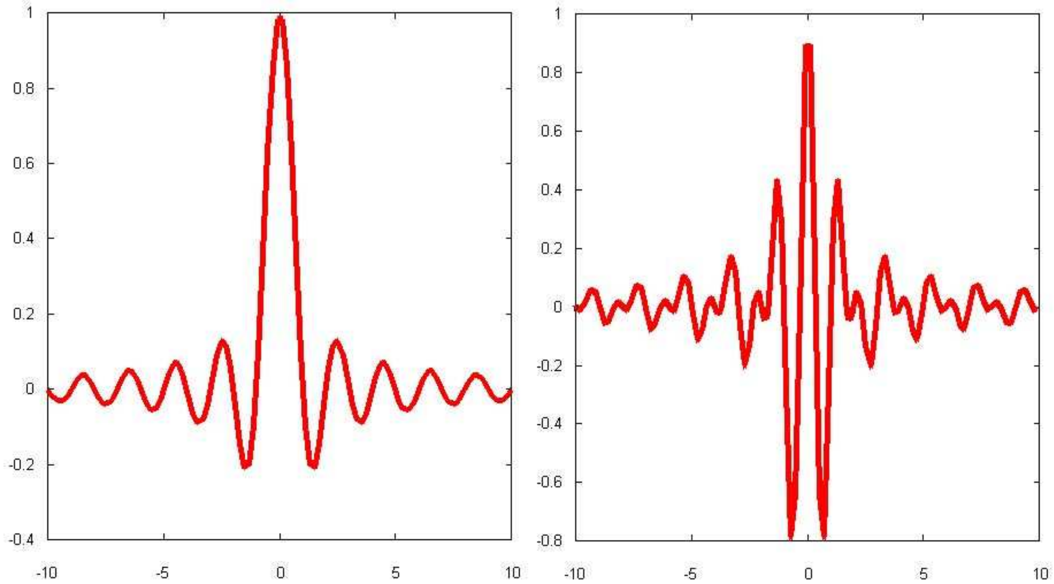

FIG. 9: Shannon scaling and wavelet function

The Fourier transform of $\varphi(x)$ is

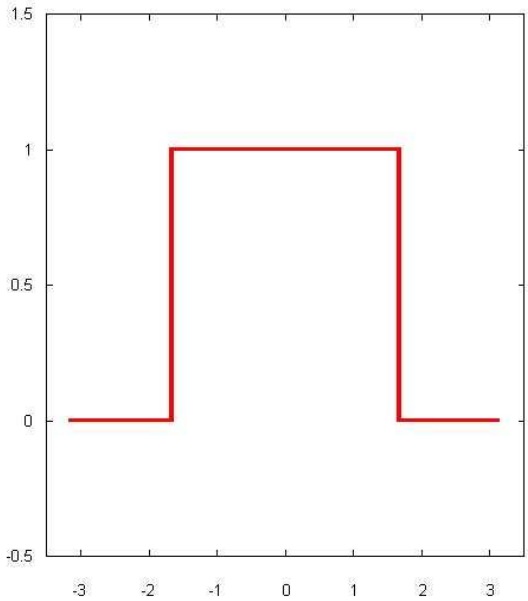

FIG. 10: Fourier transform of sinc function

It is the hat function so it has perfect localization in Fourier frequency domain. This wavelet is also called *sinc* wavelet and can be considered as a chaining link between Window Fourier Transform and Wavelet Transform. This wavelet looks similar to a wave package and is easy to handle. Another advantage is that the wavelet has infinite number of derivatives. As a disadvantage one can advance infinite support (what implies infinite number of coefficients h_i) and a very slow convergence of $\varphi(x)$ to zero when $x \to 0$.

5.5 Meyer wavelet (Y. Meyer)

It is well known than the more smooth Fourier transform of the function is the faster it decays, so to advance previous wavelet we can choose its Fourier transform as

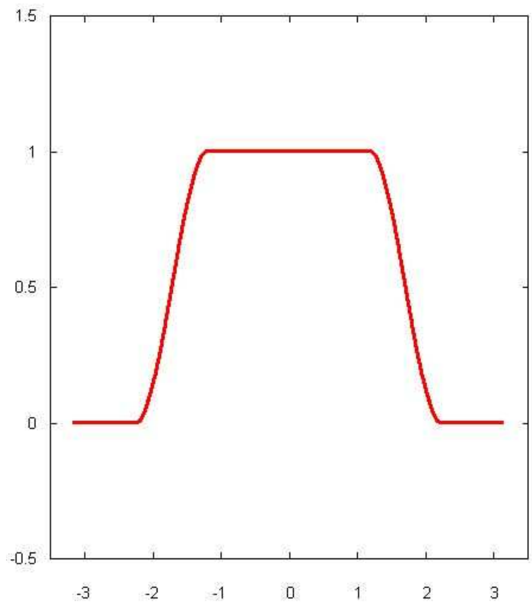

FIG. 11: Fourier transform of Meyer father function

This is the Meyer father function which decays faster but still has infinite support.

There are many other types of wavelets, and some of them listed in [1] and [2]

6 Cascade algorithm

As mentioned before there is small group of wavelets which have elementary function representation of their father functions. Nevertheless, it is very interesting and useful to know how the function looks like. To plot a function one can use an iterative algorithm based on equation (6) (It can be found in [1]).

As a trial function $\varphi_{0,0}(t)$ we choose a hat function and substitute it into

$$\varphi_{-1,k}(t) = \frac{1}{\sqrt{2}} \sum_{k \in \mathbf{Z}} h_k \varphi_{0,k}(t)$$

Let us take a look on Daubechies D4 father function construction

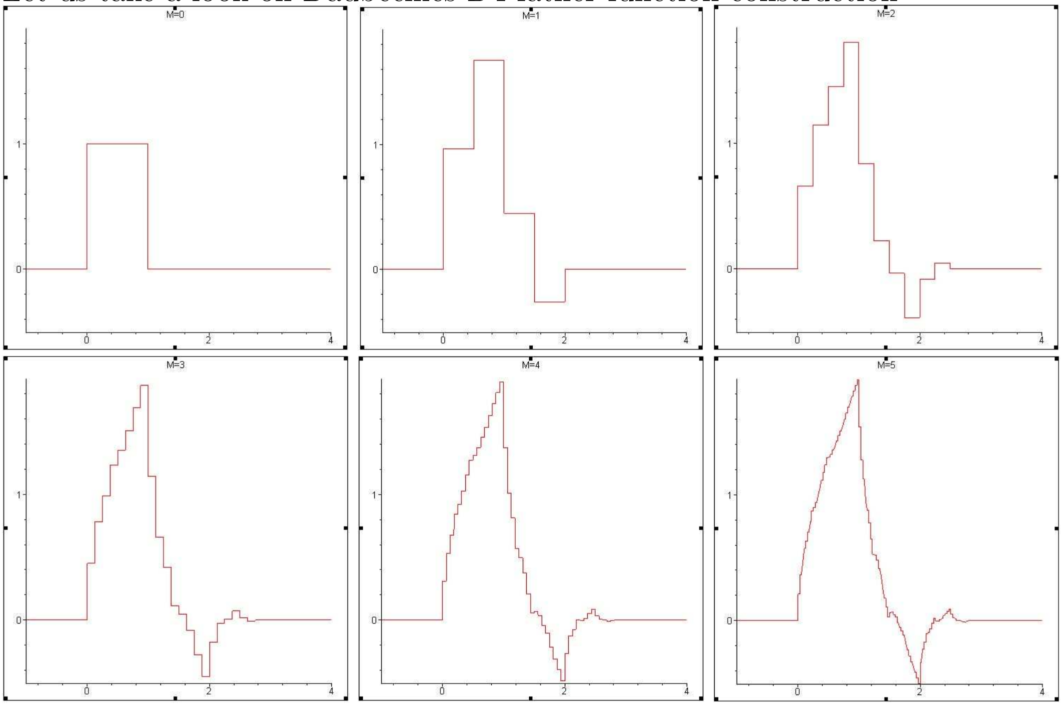

FIG. 12: First 6 iterations of the cascade algorithm for D4 father function

After several iterations one can get father function with the desired precision. Wavelet function can be plotted very similar but for the last iteration one should use (8) instead of (6).

7 Applications

The basic application of the wavelets are:

- Data processing.
- Data compression.
- Solution of differential and integral equations.

Let us see how Wavelets work for signal processing (decomposition and reconstruction) in comparison with Fourier methods.

7.1 "Digital" signal

Suppose we have a signal of the type

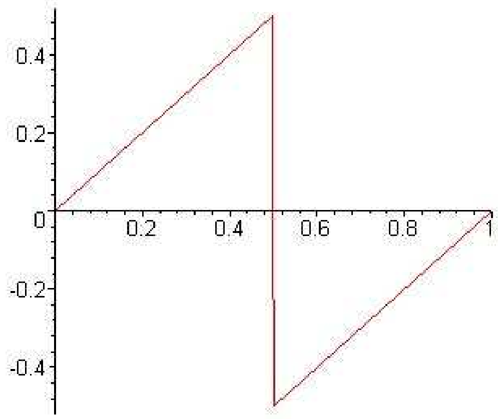

FIG. 13: "Digital" signal

let us call it "Digital" signal.
Fourier spectrum of this signal is

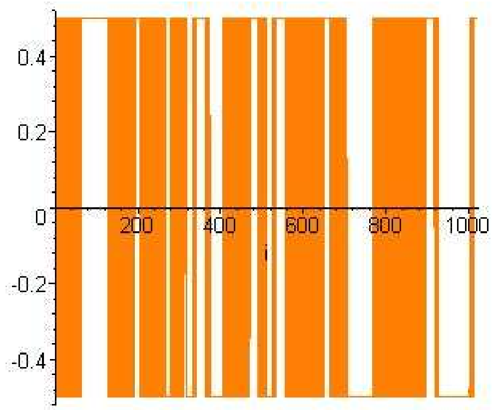

FIG. 14: Fourier transform of Digital signal

It is very difficult to understand what exactly shows this spectrum and even harder to analyze.
The "finest" 8th level coefficients are

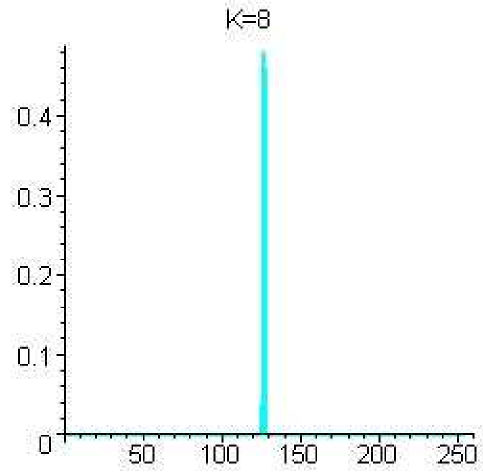

FIG. 15: Level 8 coefficients of Digital signal

Reconstructions

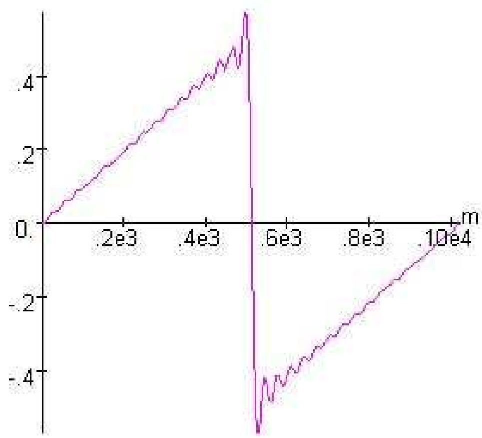

FIG. 16: Fourier reconstruction of Digital signal

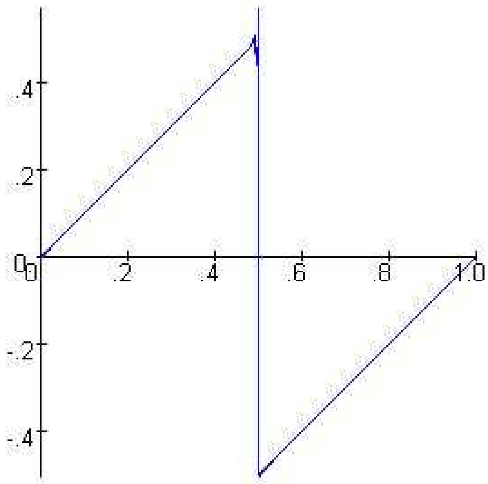

FIG. 17: Wavelet reconstruction of Digital signal

It is obvious that in this case wavelet method is superior.

7.2 "Analog" signal

Suppose we have a signal of the type

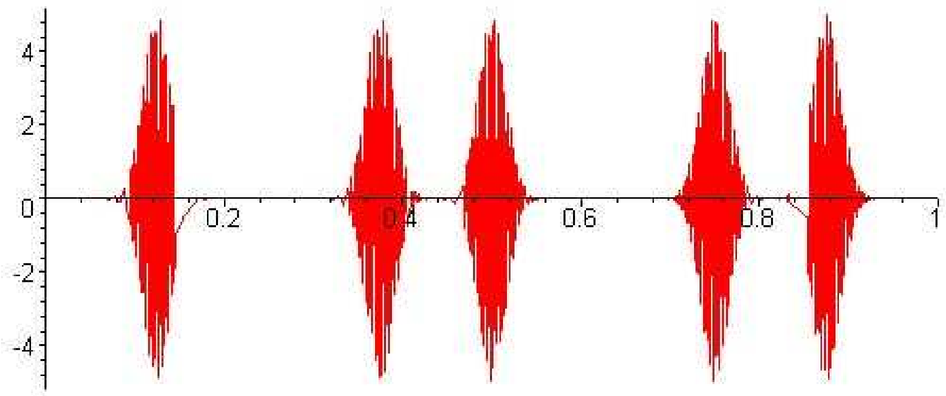

FIG. 18: "Analog" signal

It is five wave packets.
Fourier spectrum of this signal is

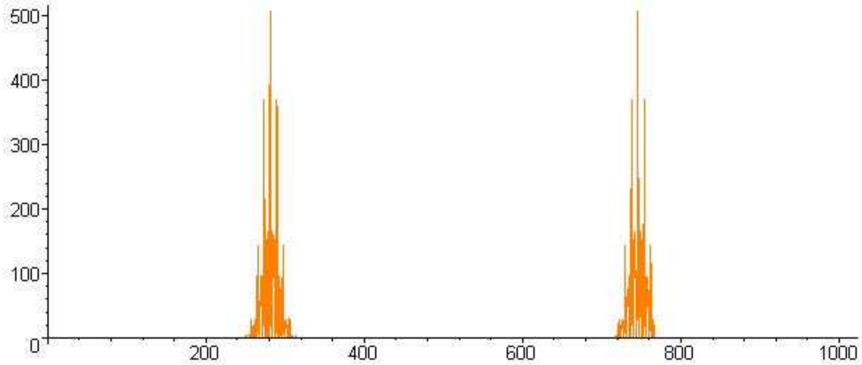

FIG. 19: Fourier transform of analog signal

It is very clear and easy to analyze.The "finest" 9th level coefficients

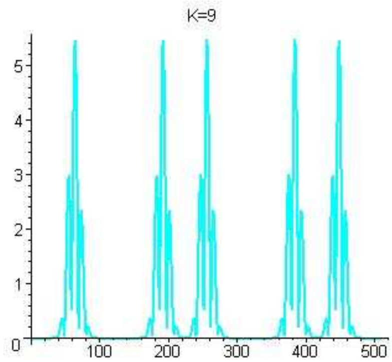

FIG. 20: Level 9 coefficients of analog signal

It is also easy to handle. Reconstructions (upper one is the Fourier reconstruction, lower one is the wavelet reconstruction)

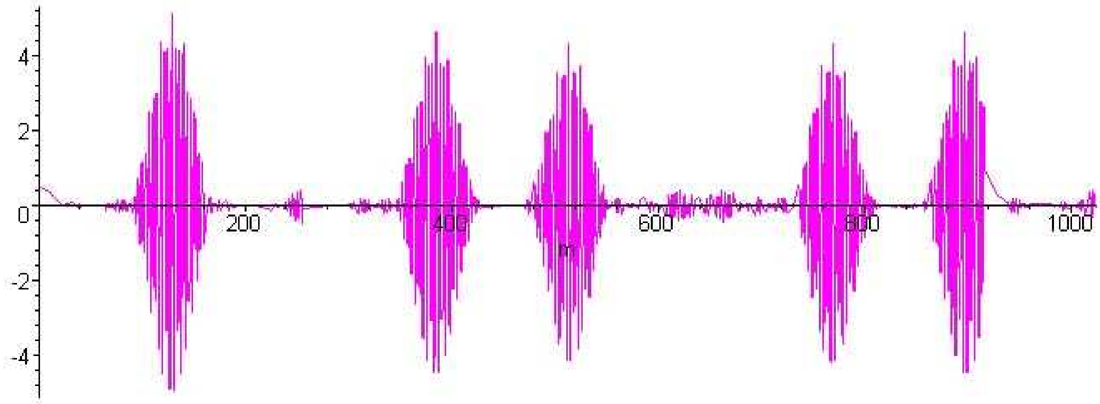

FIG. 21: Fourier reconstruction of analog signal

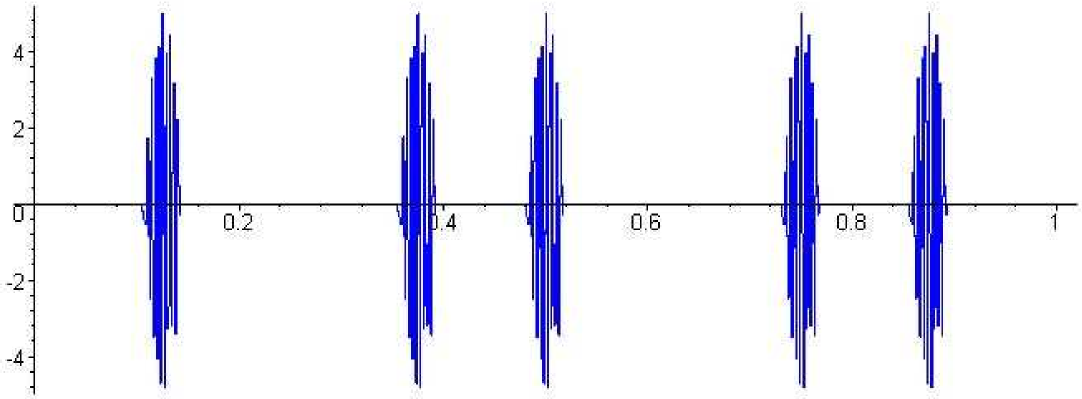

FIG. 22: Wavelet reconstruction of analog signal

Each method has its own advantages and disadvantages and it is uncertain what method to choose, so one must clearly understand what type of action will be applied next, to decide what transformation will fit the most.

7.3 Signal with short living state

Now let us choose a signal of the type

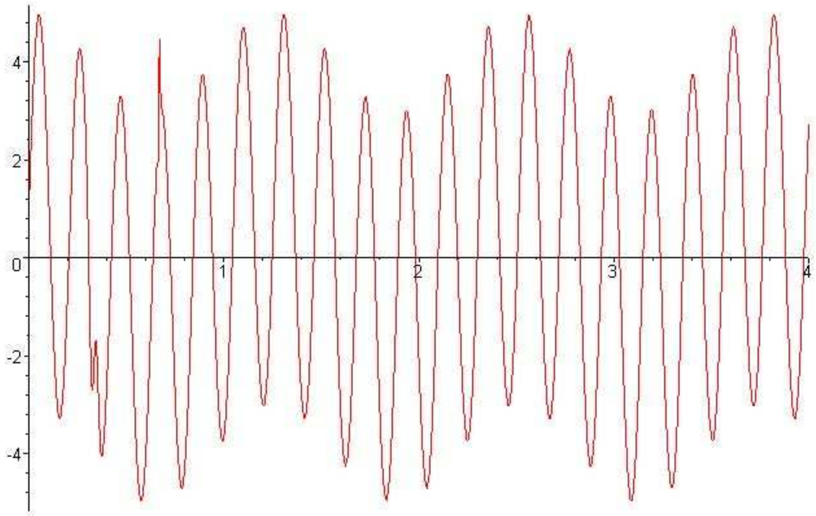

FIG. 23: Signal with short living state

Window Fourier (Gabor) transform

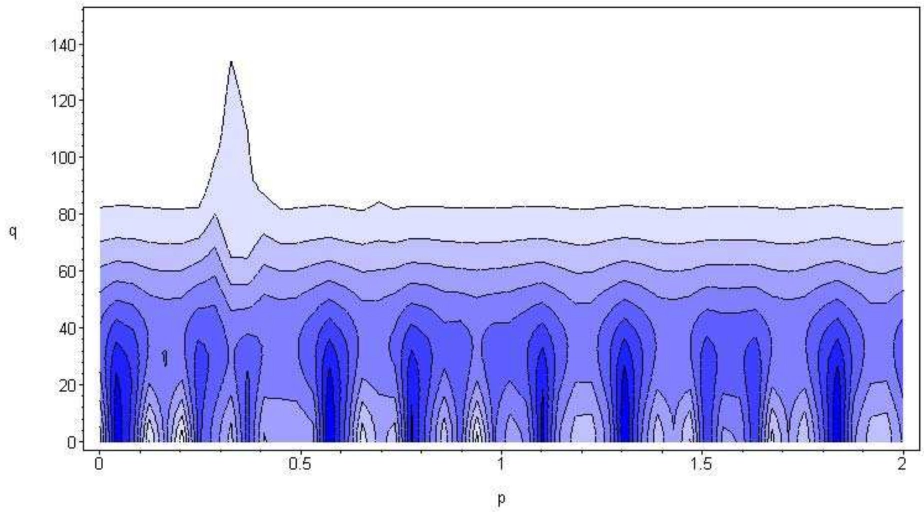

FIG. 24: Gabor transform

Wavelet transform

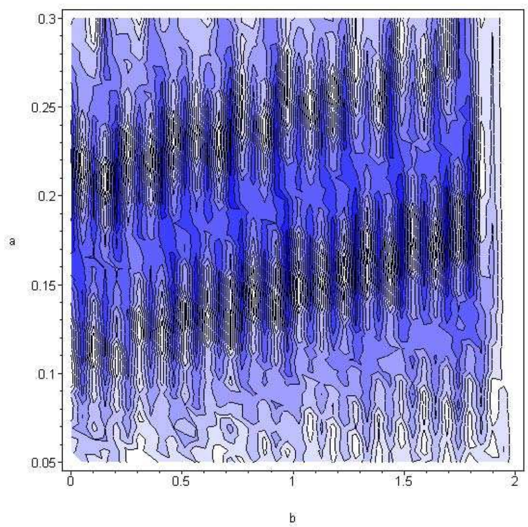

FIG. 24: Wavelet transform (Haar wavelet)

It is very difficult to find on the wavelet spectrum where is the short living state, so we can make a conclusion that in this case Gabor transform is preferable. For more applications one can refer to [4].

7.4 Conclusion

As shown before, Wavelets Transform is very useful tool for some applications. But there are some cases when wavelets could not produce any advantage in comparison with Fourier or Window Fourier methods. There are also some cases where using Fourier methods is preferable than use of wavelets. Applied to signal processing, we can make the following suggestions about the appropriate method to use

- Stationary signal — Fourier analysis,
- Stationary signal with singularities — Window Fourier analysis,
- Nonstationary signal — Wavelet analysis.

To achieve the best results one should choose carefully the proper tool for particular application. If Wavelets are chosen it is necessary to decide what type of wavelets will fit the best.

8 Acknowledgements

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