Seminar: Vibrations and Structure-Borne Sound in Civil Engineering – Theory and Applications

Analysis of Structural Vibration using the Finite Element Method

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Abstract

Structural vibration testing and analysis contributes to progress in many industries, including aerospace, auto-making, manufacturing, wood and paper production, power generation, defense, consumer electronics, telecommunications and transportation. The most common application is identification and suppression of unwanted vibration to improve product quality.

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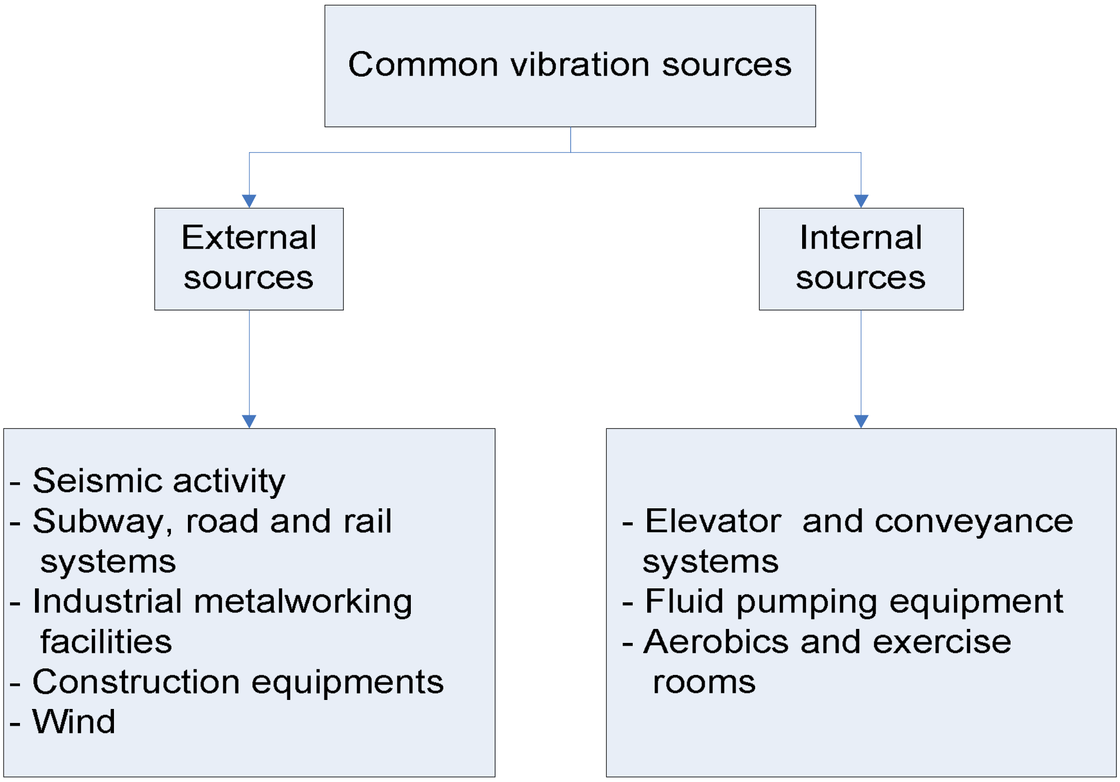
Part 1. Introduction and Basic concepts

1.1 Basic Terminology of Structural Vibration

The term vibration describes repetitive motion that can be measured and observed in a structure. Unwanted vibration can cause fatigue or degrade the performance of the structure. Therefore it is desirable to eliminate or reduce the effects of vibration. In other cases, vibration is unavoidable or even desirable. In this case, the goal may be to understand the effect on the structure, or to control or modify the vibration, or to isolate it from the structure and minimize structural response.

1.2.1. Common Vibration Sources



It can be critically important to ensure that the natural frequencies of the structural system do not match the operating frequencies of the equipment. The dynamic amplification which can occur if the frequencies do coincide can lead to hazardous structural fatigue situations, or shorten the life of the equipment.

1.2.2 Forms of Vibration

Free vibration is the natural response of a structure to some impact or displacement. The response is completely determined by the properties of the structure, and its vibration can be understood by examining the structure's mechanical properties.

Forced vibration is the response of a structure to a repetitive forcing function that causes the structure to vibrate at the frequency of the excitation.

Sinusoidal vibration is a special class of vibration. The structure is excited by a forcing function that is a pure tone with a single frequency.. The motion of any point on the structure can be described as a sinusoidal function of time 0.

Random vibration is very common in nature. The vibration you feel when driving a car result from a complex combination of the rough road surface, engine vibration, wind buffeting the car's exterior, etc.

Rotating imbalance is another common source of vibration. The rotation of an unbalanced machine part can cause the entire structure to vibrate. Examples include a washing machine, an automobile engine, shafts, steam or gas turbines, and computer disk drives.

1.3 Structural Vibration

The simplest vibration model is the **single-degree-of-freedom**, or **mass-spring-damper model**. It consists of a simple mass (M) that is suspended by an ideal spring with a known stiffness (K) and a dashpot damper from a fixed support. A dashpot damper is like a shock absorber in a car.

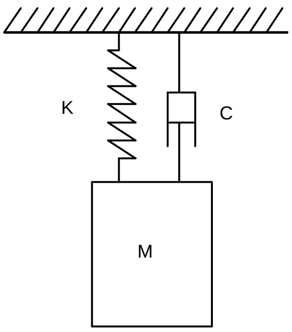


Fig.1. 3. Simple Mass-Spring-Damper vibration Model

If you displace the mass by pulling it down and releasing it, the mass will respond with motion similar to Figure 1. 4. The mass will oscillate about the equilibrium point and after every oscillation, the maximum displacement will decrease due to the damper, until the motion becomes so small that it is undetectable. Eventually the mass stops moving.

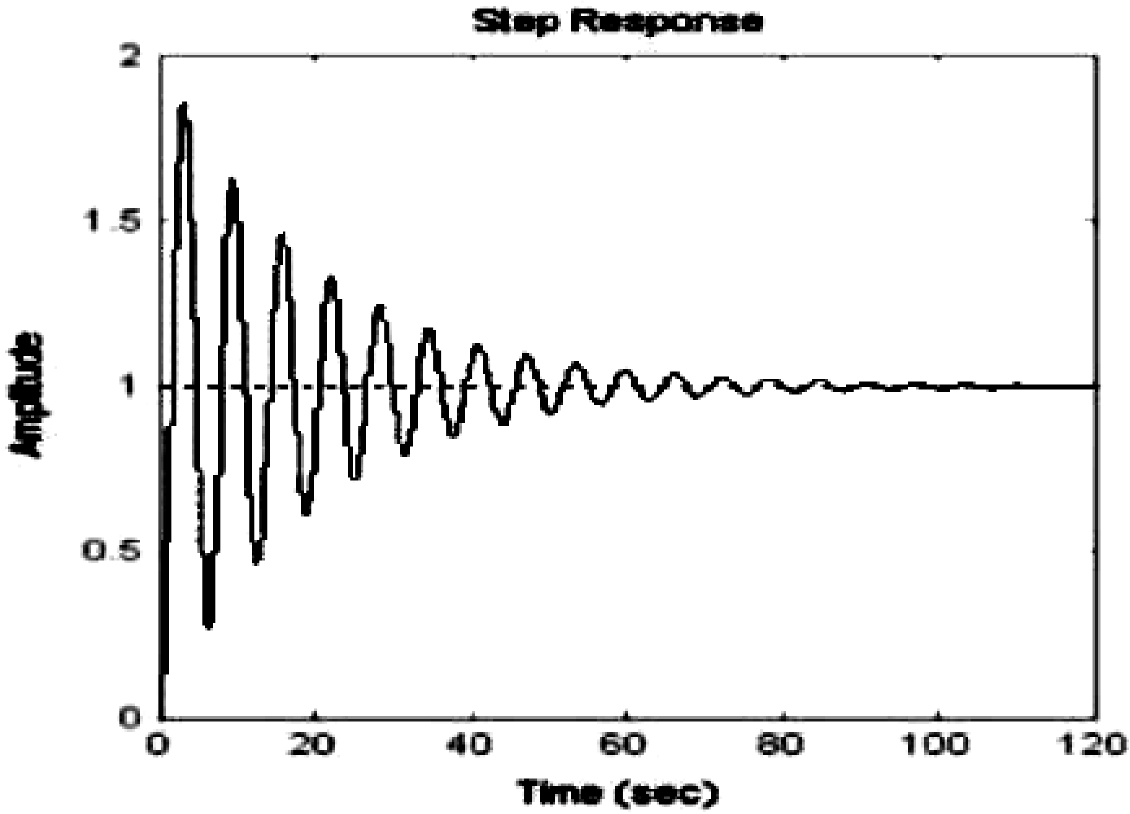


Figure 1.4. Free Vibration of Mass-Spring-Damper

Figure 1.4 shows that the time between every oscillation is the same. related to the frequency of oscillation

Assuming the damping is small, then the mathematical relationship is given by

$$w_n = \sqrt{\frac{K}{M}} \qquad (1)$$

A larger stiffness will result in a higher • $_n$, and a larger mass will result in a lower • $_n$.

Figure (1.4). also reveals something about damping. Theory tells us that the amplitude of each oscillation will diminish at a predictable rate. The rate is related to the damping factor C. Usually damping is described in terms of the damping ratio--˜. That ratio is related to C by

$$z = \frac{C}{2M\mathrm{w}_n} \tag{2}$$

When a structure is excited, it will deform, vibrate and take on different shapes depending on the frequency of the excitation and the mounting of the ends. For a continuous structure such as a beam fixed at both end and where the mass is distributed over volume. The beam will have a first resonant frequency at which all its points will move in unison; at the first resonant frequency, the beam will take the shape shown to the right in Figure (1.5) labeled First Mode Shape.

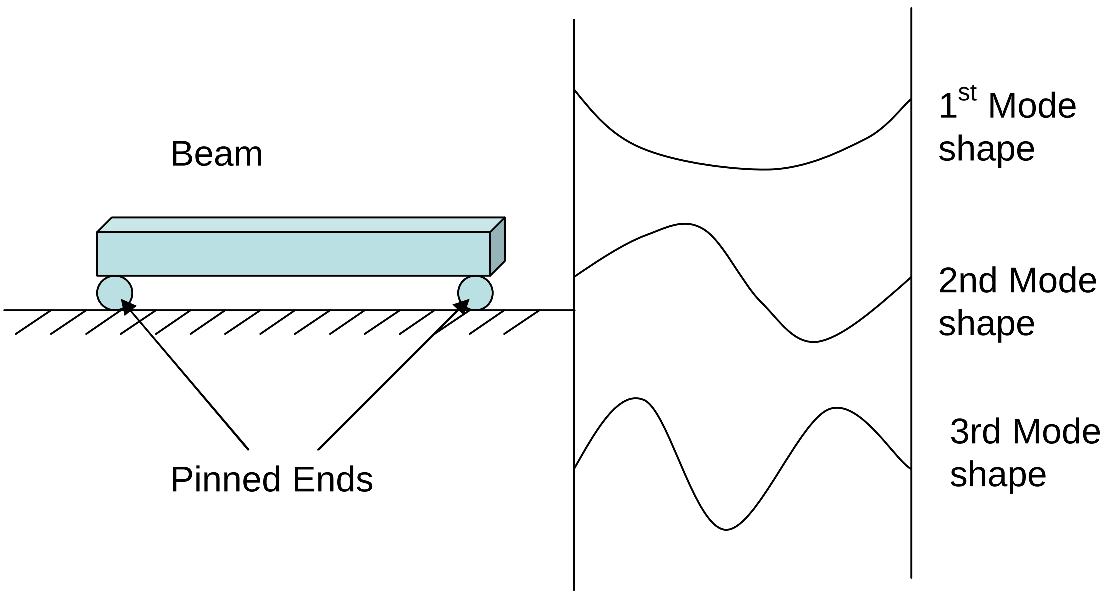


Fig.1. 5 A beam fixed at both ends.

At a higher frequency, the beam will have a second resonant frequency and mode shape, and a third, and fourth, etc. Theoretically there are an infinite number of resonant frequencies and mode shapes.

However at higher frequencies, the structure acts like a low-pass filter and the vibration levels get smaller and smaller. The higher modes are harder to detect and have less effect on the overall vibration of the structure.

Part 2. Vibration Plate.

2.1 General Theory

In three-dimensional elasticity theory the stress at a point is specified by the six quantities:

s_x, s_y, s_z – the components of direct stress;

s_{xe}, s_{yz}, s_{zx} – the components of shear stress.

The components of stress on the face *ABCD* of an element are shown in Fig. (2.1), from which the sign conventions are seen to be: direct

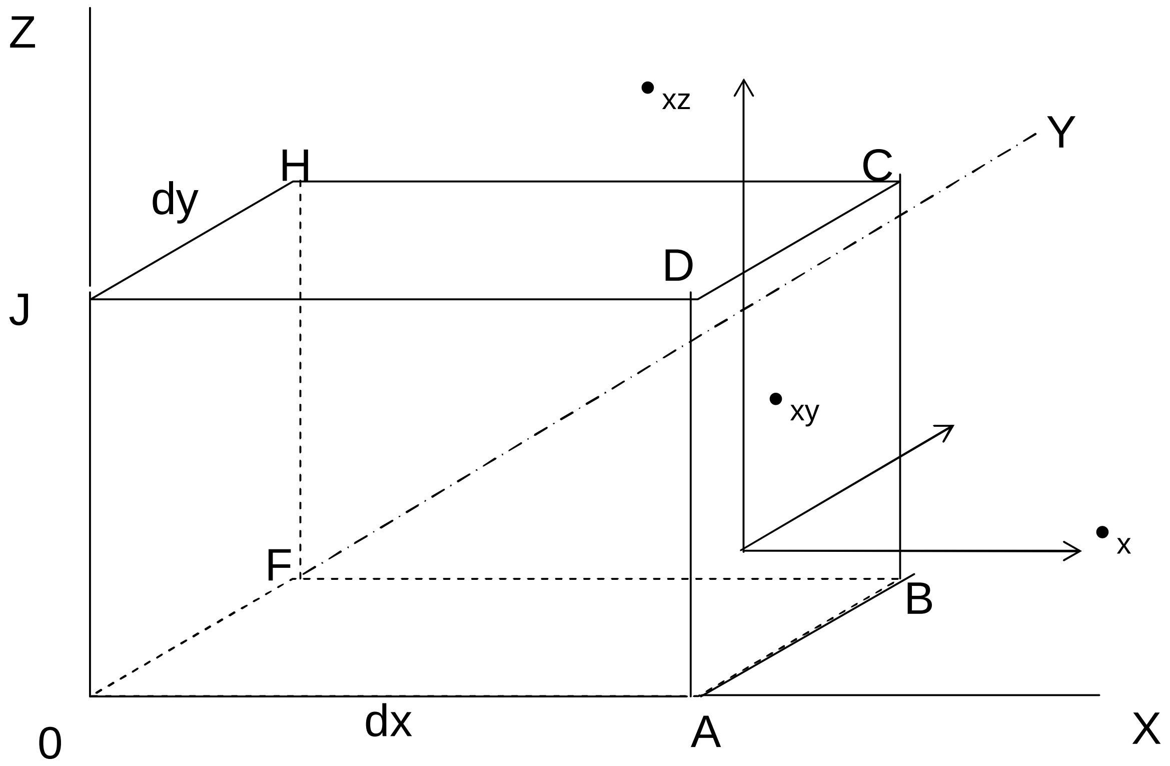


Fig. 2.1. Stress components on the face ABCD of an element

Stresses are positive when tensile; the shear stress s_{xz} acts on a face perpendicular to the X-axis in a direction parallel to the Z-axis and is positive if it acts in the positive direction of the Z-axis on a face for which the positive direct stress is in the direction If the lengths of the sides of the element are dx dy and dz, the shear stresses s_{xz} on the faces *ABCD* and *OFHJ* form a couple about the Y-axis of magnitude $(s_{xz}\ dz\ dy)\,dx$ Considering the other components of shear stress, only the components s_{zx} acting on the faces *DCHJ* and *OABF* form a couple about the Y-axis. Thus taking moments about the Y-axis for the equilibrium of the element, $s_{xz} = s_{zx}$. Similarly, $s_{xy} = s_{yx}$ and $s_{yz} = s_{zy}$ The strain at a point is defined similarly by:

$e_{x,} e_{y,} e_{z,}$ -the components of direct strain;

$e_{xy,} e_{yz,} e_{xz,}$ -the components of shear strain.

The components of displacement at any point (x, *y*, *z)* are u, *v* and w, positive in the directions OA, *OY* and *OZ*, respectively.

The components of direct stress and strain are related by Hooke's law, extended to include Poisson's ratio eff

$$e_x = \left(1/E\right)\left[s_x - v(s_y + s_z)\right] \qquad (3)$$

with analogous expressions for $e_{y,}\,and\,e_{z,}$ The components of shear stress and strain are related by

$$e_{xy} = \left(1/G\right)s_{xy,} \quad \text{etc} \qquad (4)$$

In these equations E, G and v are the elastic constants, Young's modulus, shear modulus (or modulus of rigidity) and Poisson's ratio, respectively. In this paper only homogeneous isotropic elastic solids will be considered; for these solids there are only two *independent* elastic constants and

$$E = 2G(1+v) \qquad (5)$$

Let us establish the strain-displacement relations. If the displacement in the X-direction of the point (x, y, z) is u, then the displacement in the same direction of the adjacent point $(x+dx, y, z)$ is $[u + (\partial u/\partial x)dx]$. Thus the direct strain in the X-direction

$$e_x = \frac{\text{Increase in length of element}}{\text{Initial length of element}}$$

$$= \frac{[u + (\partial u / \partial x)dx] - dx}{dx}$$

$$= \partial u / \partial x \qquad (6)$$

Similarly, $e_y == \partial v / \partial y$ and $e_z == \partial w / \partial z$

Considering an element *ABCD*, initially rectangular and having sides of length dx and dy parallel to the *X*- and F-axes, the corner *A*, initially at the point (x, y), is displaced to A_1 in the plane *OXY*, with components of displacement *u* and • in the *X*- and F-directions, respectively (Fig. 2.2). The point ˜ is displaced to B_1 with a displacement in the y-direction of $[v + (\partial v / \partial x)dx]$ the point *D* is displaced to

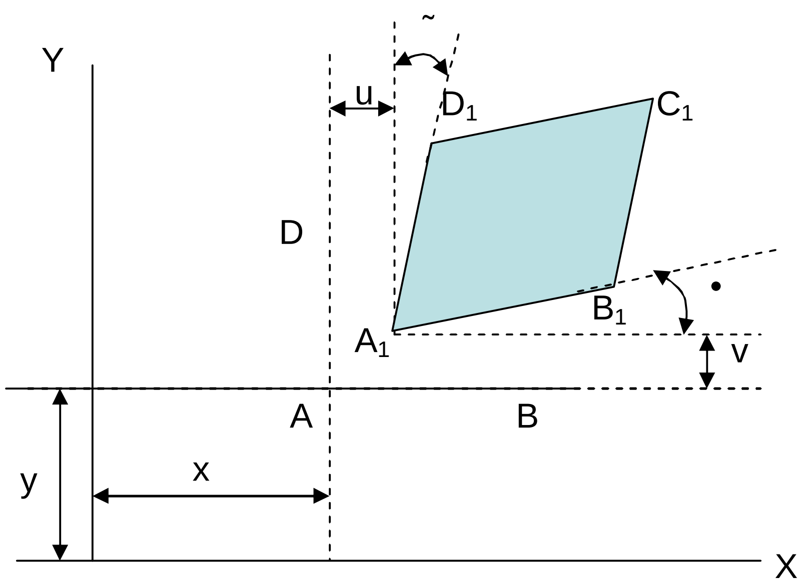


Fig. 2.2. Displacement in the plane OXY

D_1, with a displacement in the X-direction of $[u + (\partial u / \partial y)dy]$. The deformed shape of the element is the parallelogram $A_1\ B_1\ C_1\ D_1$ and the shear strain is (a + •). Thus for small angles

$$e_{xy} = a + b = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \qquad (7)$$

Similarly,

$$e_{yz} = \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} \text{ and } e_{zx} = \frac{\partial u}{\partial z} + \frac{\partial u}{\partial x}$$

For an element of sides dx, dy and dz (Fig. 5.1) the force due to the stress s_x *is* $s_x dydz$ *;* if the corresponding strain is e_x, the extension in the X-direction is $e_x dx$ and the work done on the element is

$$\frac{1}{2}(s_x dydz)(e_x dx) = \frac{1}{2} s_x e_x dV \tag{8}$$

where $dV = d_x, dy, dz$ *is* the volume of the element.

Considering the element *ABCD* of Fig. 2.2, of thickness dz in the Z-direction, the force corresponding to the shear stress s_{xy} on the face *BC* is $s_{xy} dydz$ *;* there is an equal and opposite force on the face *AD*, the two forces forming a couple of magnitude $s_{xy} dxdydz$ *;* the rotation due to this couple is •. Due to the complementary shear stress on the faces *AB* and *CD*, there is a couple of equal magnitude with corresponding rotation a. Using, $e_{xy} = a + b$, the work done on the element is

$$\frac{1}{2} s_{xy} e_{xy} dV$$

Generalizing for a three-dimensional state of stress, the strain energy in an elastic body

$$d = \int_V \frac{1}{2}(s_x e_x + s_y e_y + s_z e_z + s_{xy} e_{xy} + s_{yz} e_{yz} + s_{zx} e_{xz}) dV \tag{9}$$

Using equations (3) and (4) the strain energy may be expressed in terms of strain only or of stress only.

2.2 Transverse Vibrations of Rectangular Plate

Consider the plate element shown below Fig.(2.5). The undeformed middle plane of the plate is defined as *OX Y*, with the *X*- and Y-axes parallel to the edges of the plate; the Z-axis will be taken as positive upwards. The following assumptions are made:

1. The plate is thin and of uniform thickness *h;* thus the free surfaces of the plate are the planes

$$z = \pm \frac{1}{2} h$$

2. The direct stress in the transverse direction s_z is zero. This stress must be zero at the free surfaces, and provided that the plate is thin, it is reasonable to assume that it is zero at any section z.

3. Stresses in the middle plane of the plate (membrane stresses) are neglected, i.e. transverse forces are supported by bending stresses, as in flexure of a beam. For membrane action not to occur, the displacements must be small compared with the thickness of the plate.

4. Plane sections that are initially normal to the middle plane remain plane and normal to it. A similar assumption was made in the elementary theory for beams and implies that deformation due to transverse shear is neglected. Thus with this assumption the shear strains s_{xz} and s_{yz} are zero.

5. Only the transverse displacement *w* (in the Z-direction) has to be considered.

Figure (2.3a) shows an element of the plate of length *dx* in the unstrained state and Fig. 2.3b the corresponding element in the strained state. If *OA* represents the middle surface of the plate, then *OA* = ˜*1* •*1* from the third assumption; also ˜*1* •*1* $= R_x dq$, where R_x is the radius of curvature of the deformed middle surface. Thus the strain in *BC*, at a distance z from the middle surface,

$$e_x = \frac{B_1 C_1 - BC}{BC} = \frac{(R_x + z) - R_x}{R_x} = \frac{z}{R_x}$$

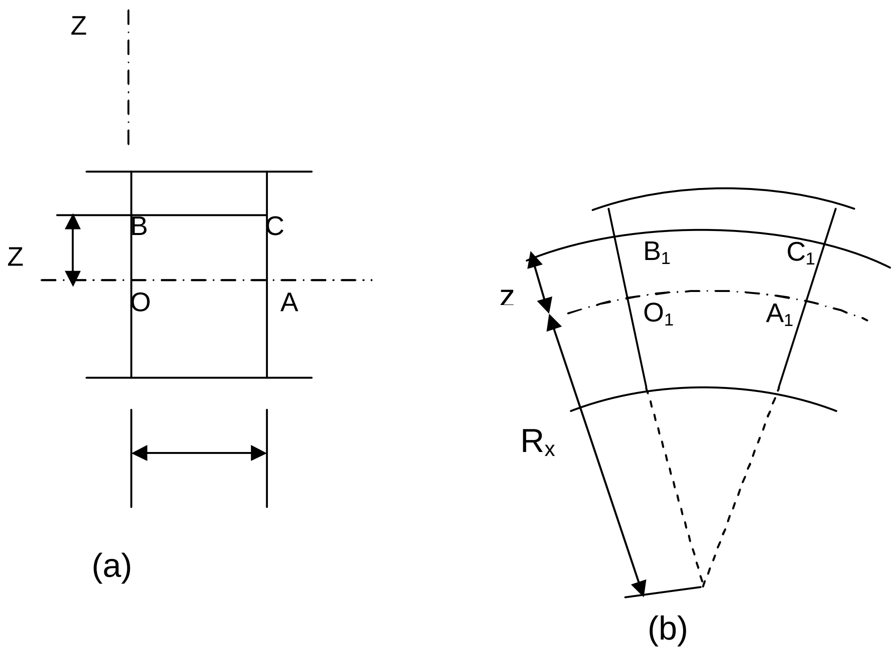


Fig.2.3. Element of plate. (a) Unstrained. (b) Strained

The relation between the curvature and the displacement of the middle surface, w, is:

$$\frac{1}{R_x} = -\frac{\partial^2 w}{\partial x^2}$$

Thus $$e_x = -z\frac{\partial^2 w}{\partial x^2} \qquad (10)$$

Similarly $$e_y = -z\frac{\partial^2 w}{\partial y^2}$$

From equation (7) the shear strain e_{xy}, at a distance z from the middle surface is

$$\left[\left(\frac{\partial u}{\partial y}\right)+\left(\frac{\partial v}{\partial x}\right)\right],$$

where u and v are the displacements at depth z in the X- and Y-directions, respectively. Using the assumption hat sections normal to the middle plane remain normal to it,

$$u = -\frac{\partial w}{\partial x} \quad . \qquad \text{Similarly} \quad v = -\frac{\partial w}{\partial y}$$

Thus $e_{xy} = -2z\dfrac{\partial^2 w}{\partial x \partial y}$ (11)

(In equation (11) the term $\dfrac{\partial^2 w}{\partial x \partial y}$ is the twist of the surface.)

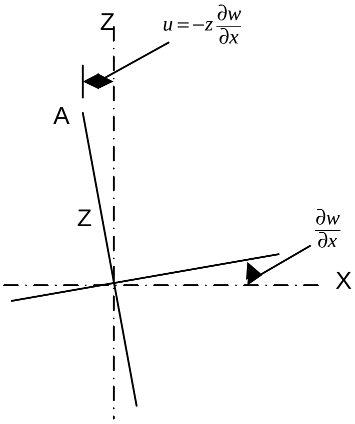


Fig. 2.4 Relation between u and w

From the second assumption and equations (3) and (4), the stress-strain relations for a thin plate are

$$s_x = \frac{E}{1-v^2}\left(e_x + v e_y\right);\ s_y = \frac{E}{1-v^2}\left(e_y + v e_x\right) \qquad (12)$$

Also $s_{xy} = \dfrac{E}{2(1-v)} e_{xy}$

Substituting from equations (10), (11) and (12) in the strain energy expression (9),

$$d = \int_0^a \int_0^b \int_{-h/2}^{h/2} \frac{E}{2(1-v^2)}\left[e^2{}_x + e^2{}_y + 2v e_x e_y + \frac{1}{2}(1-v^2) e^2{}_x e_y\right] dzdydx$$

$$= \frac{D}{2}\int_0^a \int_0^b \left[\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + \left(\frac{\partial^2 w}{\partial y^2}\right)^2 + 2v\left(\frac{\partial^2 w}{\partial x^2}\right)\left(\frac{\partial^2 w}{\partial y^2}\right) + 2(1-v)\left(\frac{\partial^2 w}{\partial x^2}\right)^2\right] dydx \qquad (13)$$

after integrating with respect to z, where

$$D = \frac{Eh^3}{12\left(1-v^2\right)}$$

The limits in the above integrals imply that the plate is bounded by the lines

$x = 0, x = a, y = 0$ and $y = b.$

If • is the density of the plate, the kinetic energy

$$\Im = \int_0^a\int_0^b\int_{-h/2}^{h/2} \frac{1}{2}\, \text{r} \left(\frac{\partial w}{\partial t}\right)^2 dzdydx$$

$$= \frac{1}{2}\, \text{r} \int_0^a\int_0^b \left(\frac{\partial w}{\partial t}\right)^2 dydx \qquad (14)$$

after integrating with respect to z. Expressions (13) and (14) will be used with the finite element methods.

The element of the plate with sides *dx* and *dy* and thickness *h,* shown in Fig. 2.5, is subjected to a bending moment M_x a twisting moment M_{xy} and a transverse shear force S_x per unit length on the face *OB;* on the face ˜*A* here are, per unit length, a bending moment M_y, a twisting moment M_{yx} and a shear force S_x.

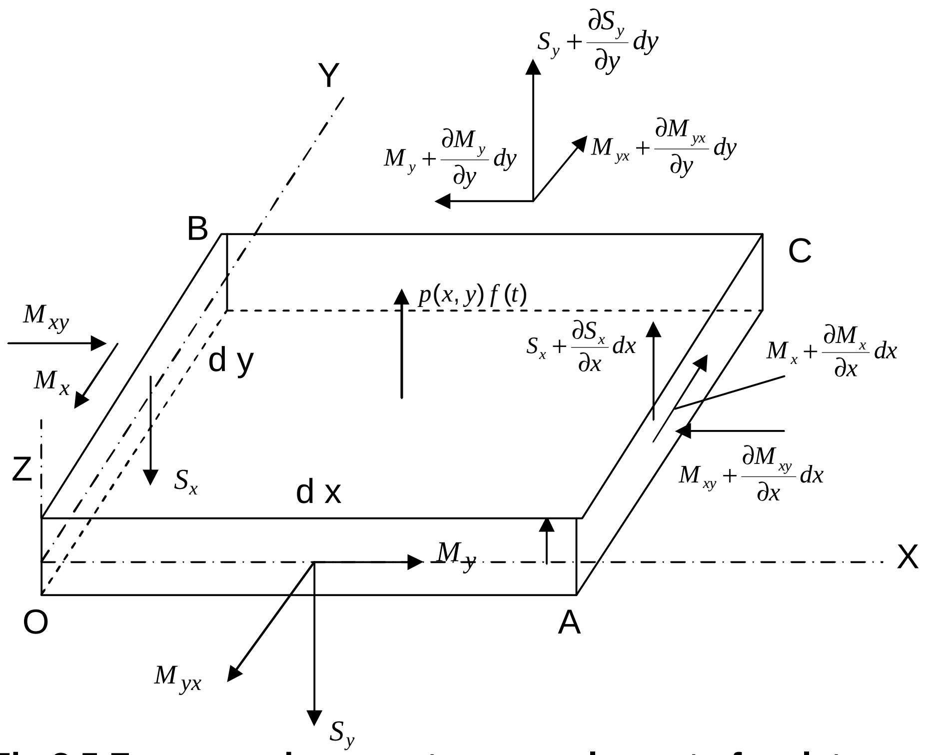


Fig.2.5 Forces and moments on an element of a plate

The bending moments M_x and M_y are the resultant moments due to the direct stresses σ_x and σ_y respectively, after integrating through the thickness of the plate. Similarly, the twisting moments are the resultants due to the shear stress σ_{xy}. Maintaining a consistent sign convention between the definitions for stresses in Fig. 2.1 and the moments shown in Fig. 2.5, we have

$$M_x = \int_{-h/2}^{h/2} \sigma_x z dz \, ; \qquad M_y = \int_{-h/2}^{h/2} \sigma_y z dz$$

and

$$M_{xy} = M_{yx} = \int_{-h/2}^{h/2} \sigma_{xy} z dz \tag{15}$$

Figure 2.5 shows also the incremental quantities acting on the faces AC and BC and the applied force per unit area, p(x, *y) f (t),* in the Z-direction. In addition there is an inertia force per unit area, $\rho h \frac{\partial^2 w}{\partial t^2}$, in the Z-direction. The equilibrium equations, obtained by resolving in the Z-direction and taking moments about the *Y*- and X-axes, are, after dividing by *dx dy:*

$$\frac{\partial S_x}{\partial x} + \frac{\partial S_y}{\partial y} + p(x,y) f(t) = \rho h \frac{\partial^2 w}{\partial t^2}$$

$$\frac{\partial M_x}{\partial x} + \frac{\partial M_{yx}}{\partial y} - S_x = 0 \quad \text{and} \quad -\frac{\partial M_y}{\partial y} + \frac{\partial M_{xy}}{\partial x} + S_y = 0$$

Eliminating S_x and S_y

$$\frac{\partial^2 M_x}{\partial x^2} + \frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} + p(x,y) f(t) = \rho h \frac{\partial^2 w}{\partial t^2} \tag{16}$$

Substituting from equations (10) to (12) in equations (15) and integrating with respect to z.

$$M_x = -D\left(\frac{\partial^2 w}{\partial x^2} + v\frac{\partial^2 w}{\partial y^2}\right); \qquad M_y = -D\left(\frac{\partial^2 w}{\partial y^2} + v\frac{\partial^2 w}{\partial x^2}\right),$$

and

$$Mxy = -D(1-v)\frac{\partial^2 w}{\partial x \partial y} \tag{17}$$

Substituting from equations (17) in equation (16) gives the equilibrium equation for an element of the plate in terms of w and its derivatives,

$$D\left[\frac{\partial^4 M_x}{\partial x^4} + \frac{\partial^4 M_{xy}}{\partial x^2 \partial y^2} + \frac{\partial^4 M_y}{\partial y^4}\right] + \rho h \frac{\partial^2 w}{\partial t^2} = p(x, y) f(t) \tag{18}$$

For a dynamic problem $w(x, y, t)$ must satisfy equation (18) together with the boundary conditions.

The standard simple boundary conditions are simply supported, clamped and free.

Part 3. Application of Finite Element Method.

3.1 Finite Element Method.

The basic concept of the finite element approach is to subdivide a large complex structure into a finite number of simple elements, such as beam elements, quadrilaterals, or triangles and the complex differential equations are then solved for the simple elements. Assemblage of the elements into a global matrix transforms from a differential equations formulation over a continuum to a linear algebra problem that is readily solvable by using computers..

In static analysis. The finite element method solves equations of the form

$$[K]\{U\} = \{P\} \tag{19}$$

where [K] is a global stiffness matrix, {U} is a vector of nodal displacements, and {P} is a vector of nodal forces.

The dynamic analysis usually starts with an equation of the form

$$\left([K] - \omega^2 [M]\right)\{U\} = \{P\} \tag{20}$$

where • in this equation denotes the frequencies of vibration and [M] is the mass matrix. For a free vibration, we have {P} = 0 and Eq. 20. reduces to the following expression:

$$\left([K] - \omega^2 [M]\right)\{U\} = \{0\} \tag{21}$$

For a nontrivial solution, we have

$$[K] = w^2 [M] \tag{22}$$

which represents the generalized Eigenvalue problems.

3.2 Element Stiffness and Mass Matrices

3.2.1 Axial Elements

Consider a truss element shown in Fig. (2.6), which has the degree of freedom u_1 and u_2 at points 1 and 2 , respectively. Since the members of the truss are loaded axially, the element would be subject to axial forces f_1 and f_2 as shown in the figure, and consequently, u_1 and u_2 would represent axial displacements. We can assume here that the displacement function u(x) may be represented by the following linear equation:

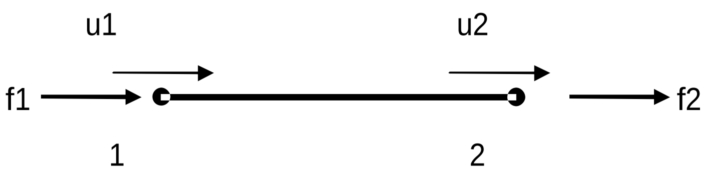


Figure (2.6). Truss element.

$$u(x) = b_1 + b_2 x \tag{23}$$

where b_1 and b_2 are constants. In matrix form Equation (23) may be written as follows:

$$u(x) = \{1 \quad x\} \begin{Bmatrix} b_1 \\ b_2 \end{Bmatrix} \tag{24}$$

By using Eq. (23) and applying the boundary conditions u(0) = u_1 and u(L) = u_2, where L is the length of the elements, we find

$$u(0) = b_1 + (0) b_2 \tag{25}$$

$$u(L) = b_2 + L b_2 \tag{26}$$

Equations (25) and (26) are written in matrix form as follows

$$\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & L \end{bmatrix} \begin{Bmatrix} b_1 \\ b_2 \end{Bmatrix} \tag{27}$$

The same two equations may also be written in a matrix form that relates b_1 and b_2 and u_1 and u_2 in the following way

$$\begin{Bmatrix} b_1 \\ b_2 \end{Bmatrix} = \begin{bmatrix} L & 0 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \tag{28}$$

By substituting Eq. (28) into Eq. (24), we find

$$u(x) = \begin{Bmatrix} 1 - \dfrac{x}{L} & \dfrac{x}{L} \end{Bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \tag{29}$$

or, in general

$$u(x) = \{H_1 \quad H_2\} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \tag{30}$$

where H, in Eq. (30) are referred to as shape functions or interpolation functions

The axial strain e of the element in Fig. (2.7) is the rate of change of $u(x)$ with respect to x. Therefore, by differentiating Eq. (29) with respect to x, we find

$$e = \begin{Bmatrix} -\dfrac{1}{L} & \dfrac{1}{L} \end{Bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \tag{31}$$

By applying Hooke's law, the normal stress ˜ at cross sections along the length of the element is given by the expression

$$s = Ee \tag{32}$$

where • is Young's modulus of elasticity and ˜ is given by Eq. (31).

The total energy Π stored in the element is defined by the expression

$$\Pi = \frac{1}{2}\int_0^L EA\left(\frac{du}{dx}\right)^T\left(\frac{du}{dx}\right)dx - f_1u_1 - f_2u_2 \tag{33}$$

where A is the cross-sectional area of the member. By using Eq. (30), the expression given by Eq. (33) may be written as follows:

$$\Pi = \frac{1}{2}\int_0^L EA\{u_1 \quad u_2\}\begin{pmatrix} -\dfrac{1}{L} \\ \dfrac{1}{L} \end{pmatrix}\begin{pmatrix} -\dfrac{1}{L} & \dfrac{1}{L} \end{pmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}dx - \{f_1 \quad f_2\}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \tag{34}$$

When • A is constant, we have

$$\int_0^L dx = L \tag{35}$$

and equation 34. yields

$$\Pi = \frac{EAL}{2}\{u_1 \quad u_2\}\begin{bmatrix} \frac{1}{L^2} & -\frac{1}{L^2} \\ -\frac{1}{L^2} & \frac{1}{L^2} \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} - \{f_1 \quad f_2\}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$= \frac{1}{2}\{u_1 \quad u_2\}\frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} - \{f_1 \quad f_2\}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \tag{36}$$

For a stationary condition, we have

$$[K] = \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \tag{39}$$

In a similar manner, various other element stiffness matrices may be obtained.

3.3 Finite Element Method for In-plane Vibration of Plate

In the conventional theory of plate vibration (Section 2.2) the stresses in the middle plane of the plate are neglected. This is a valid assumption for small vibrations of uniform plates. However, for a plate reinforced by eccentric stiffeners or for a system of plates, built up to represent approximately a curved or shell structure, bending and membrane, or in-plane, deformations are coupled.

In section finite elements for in-plane and vibrations will be developed. The elements for in-plane vibrations are chosen, as they are simpler in concept.

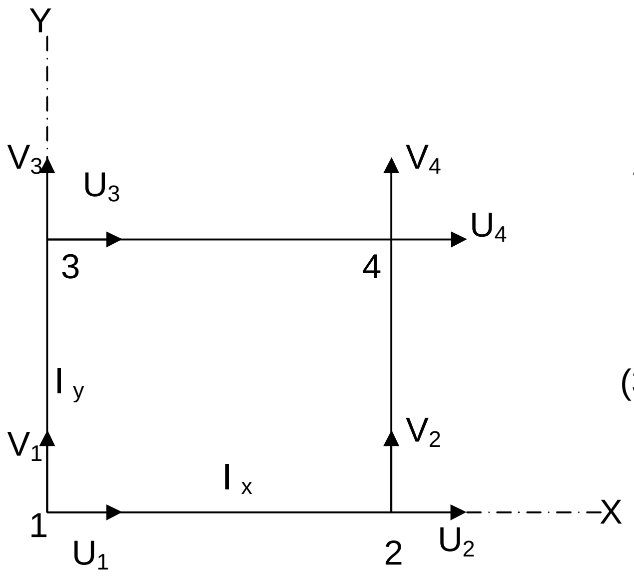


Fig.2.7 Rectangular plate for in-plane deformation

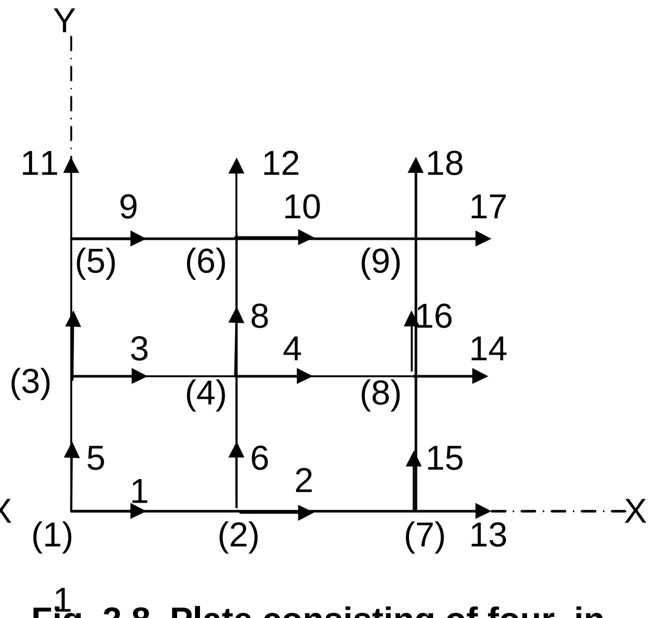


Fig. 2.8 Plate consisting of four in-plane deformation

Figure 2.7 shows a rectangular element of sides l_x and l_y. The middle surface of the element lies in the plane *OXY*. We consider a state of plane stress; i.e. the non-zero components of stress are σ_x, σ_y and σ_{xy}. As the stresses are assumed to be uniform through the thickness h, the strain energy of the element

$$\delta_e = \frac{1}{2} h \int_0^{l_y}\int_0^{l_x} (\sigma_x \varepsilon_x + \sigma_y \varepsilon_y + \sigma_{xy} \varepsilon_{xy}) dxdy \tag{40}$$

from equation (9). If u and v are the displacements in the *X*- and *Y*-directions, respectively, the strain energy can be expressed in terms of displacements, using the stress-strain relations (12) and the strain-displacement relations (7) and (8),

$$\delta_e = \frac{1}{2}\frac{Eh}{1-v^2}\int_0^{l_y}\int_0^{l_x} \left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + 2v\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} + \frac{(1-v)}{2}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)^2\right] dxdy$$

$$=\frac{1}{2}\int_0^{l_y}\int_0^{l_x}\left[\frac{\partial u}{\partial x},\frac{\partial v}{\partial y},\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)\right]D\begin{bmatrix}\frac{\partial u}{\partial x}\\ \frac{\partial v}{\partial y}\\ \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\end{bmatrix}dxdy \qquad (41)$$

The element of Fig. 2.7s has nodes at the four corners; thus nodal variables u_j are v_j and with j = 1, 2, 3 and 4. The assumed displacement functions are

$$u = a_1 + a_2 x + a_3 y + a_4 xy$$

$$v = a_5 + a_6 x + a_7 y + a_8 xy \qquad (42)$$

or

$$\begin{bmatrix}u\\ v\end{bmatrix}=\begin{bmatrix}g & 0\\ 0 & g\end{bmatrix}a$$

where $g=[1 \quad x \quad y \quad xy]$ and a is a vector containing the coefficients $a_1, a_2, ...a_8$. Substituting the nodal values

$$u_e=\begin{bmatrix}u_1\\u_2\\u_3\\u_4\\v_1\\v_2\\v_3\\v_4\end{bmatrix}=Na=\begin{bmatrix}1&0&0&0&0&0&0&0\\1&l_x&0&0&0&0&0&0\\1&0&l_y&0&0&0&0&0\\1&l_x&l_y&l_xl_y&0&0&0&0\\0&0&0&0&1&0&0&0\\0&0&0&0&1&l_x&0&0\\0&0&0&0&1&0&l_y&0\\0&0&0&0&1&l_x&l_y&l_xl_y\end{bmatrix}\begin{bmatrix}a_1\\a_2\\a_3\\a_4\\a_5\\a_6\\a_7\\a_8\end{bmatrix} \qquad \mathbf{(43)}$$

Thus

$a = Bu_e$ where $B = N^{-1}$

As $\partial u/\partial x = a_2 + a_4 y, \quad \partial v/\partial y = a_7 + a_8 y$

and $\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)=a_3+a_4x+a_6+a_8y$

$$\begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y} \end{bmatrix} = Ga \quad \text{where} \quad G=\begin{bmatrix} 0 & 1 & 0 & y & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & x \\ 0 & 0 & 1 & x & 0 & 1 & 0 & y \end{bmatrix} \tag{44}$$

Substituting from equation (44) in equation (41)

$$\mathrm{d}_e = \frac{1}{2}\int_0^{l_y}\int_0^{l_x} a^T G^T DGadxdy$$

$$=\frac{1}{2}u^T{}_e B^T\left[\int_0^{l_y}\int_0^{l_x} G^T DGdxdy\right]Bu_e$$

$$=\frac{1}{2}u^T{}_e K_e u_e$$

where the element stiffness matrix

$$K_e = B^T\left[\int_0^{l_y}\int_0^{l_x} G^T DGdxdy\right]B \tag{45}$$

The kinetic energy of the element

$$\Im_e = \frac{1}{2}\,\mathrm{r}\,h\int_0^{l_y}\int_0^{l_x}\left[\left(\frac{\partial u}{\partial t}\right)^2+\left(\frac{\partial v}{\partial t}\right)^2\right]dydx$$

$$=\frac{1}{2}\int_0^{l_y}\int_0^{l_x}\begin{bmatrix}\frac{\partial u}{\partial t} & \frac{\partial v}{\partial t}\end{bmatrix}\mathrm{r}\,h\begin{bmatrix}\frac{\partial u}{\partial t} \\ \frac{\partial v}{\partial t}\end{bmatrix}dydx$$

$$=\frac{1}{2}\dot{u}^T_e M_e \dot{u}_e \tag{46}$$

where the element mass matrix

$$M_e = B^T \rho h \int_0^{l_y}\int_0^{l_x} \begin{bmatrix} g^T g & 0 \\ 0 & g^T g \end{bmatrix} dydxB \tag{47}$$

Before considering the assembly of elements to represent the structure the conditions to be satisfied by the assumed displacement functions (42) will be discussed. These will be presented in general terms, so that they are applicable to other types of element.

1. Displacements and their derivatives up to the order one less than that occurring in the strain energy expression should be continuous across element boundaries.
2. The displacement functions should be able to represent appropriate rigid-body motions.
3. The displacement functions should be able to represent states of constant strain.

If these conditions are satisfied, we have conforming or compatible elements; for eigenvalue calculations with conforming elements representing a structure the eigenvalues converge monotonically from above to the correct values with progressive subdivision of the element mesh. For non-conforming elements with condition (3) satisfied eigenvalues converge to the correct values eventually, as the mesh is refined, but convergence is not monotonic. If condition (3) is not satisfied, the elements are too stiff and eigenvalues converge on values higher than the correct ones.

3.4 Finite Element Method for Transverse Vibrations of Plates

Figure (2.9) shows a rectangular element with sides of length l_x and l_y and having nodes at the four corners. The nodal variables are the transverse displacement w_j and the rotations $\phi_j\ (\equiv -\partial w_j/\partial x)$ and $\psi_j\ (\equiv -\partial w_j/\partial y)$ with j ; = 1, 2, 3 and 4. (For a right-hand set of axes the Z-axis in Fig. (2.9) is outwards from the plane of the diagram. The displacement w is in the Z-direction; ϕ and ψ are positive (i.e. clockwise) rotations about the *Y*- and X-axes, respectively. These definitions cause the negative sign in the relation between ϕ and $\partial w/\partial x$

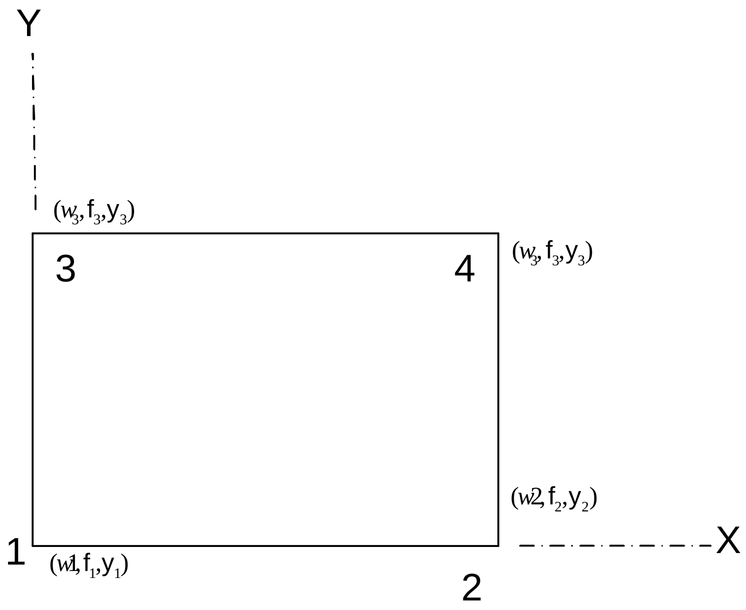


Fig.2.9 Rectangular plate for flexure of a plate

The assumed displacement function is

$$w = a_1 + a_2 x + a_3 y + a_4 x^2 + a_5 xy + a_6 y^2 + a_7 x^3 + a_8 x^2 y + a_9 xy^2 + a_{10} y^3 + a_{11} x^3 y + a_{12} xy^3 = ga$$

(48)

Where **g** is a row matrix of polynomial terms and the vector **a** contains the twelve coefficients a_i. The term in a_1 ensures rigid-body translation; those in a_2 and a_3 ensure rigid-body rotations; those in a_4 and a_6 ensure states of uniform curvature; and that in a_5 ensures a state of uniform twist. Along *OX*, the line joining nodes 1 and 2 in Fig. 2.9, putting y = 0 in equation (48),

$$w = a_1 + a_2 x + a_4 x^2 + a_7 x^3$$

$$\text{f} = -\left[a_2 + 2a_4 x + 3a_7 x^2 \right]$$

Thus the coefficients a_1, a_2, a_4 and a_7 are uniquely defined in terms of the four nodal values w_1, f_1, w_2 and f_2. As the latter are common to the two elements, for which *OX* is a common boundary, there is continuity of *w* and • across the inter-element boundary. Also along *OX*,

$$\text{y} = a_3 + a_5 x + a_8 x^2 + a_{11} x^3$$

Substituting nodal values of x and y in equation (48),

$$w_e = \begin{bmatrix} w_1 \\ w_2 \\ . \\ . \\ . \\ \mathrm{f}_1 \\ \mathrm{f}_2 \\ . \\ . \\ . \\ \mathrm{y}_1 \\ \mathrm{y}_2 \\ . \\ . \\ . \end{bmatrix} = Na$$

and

$$a = N^{-1} w_e = B w_e \tag{49}$$

From equation (13) the strain energy expression for the element can be written

$$\mathrm{d}_e = \frac{1}{2} \int_0^{l_y} \int_0^{l_x} \left[\frac{\partial^2 w}{\partial x^2}, \frac{\partial^2 w}{\partial y^2}, \left(\frac{2\partial^2 w}{\partial x \partial y} \right) \right] C \begin{bmatrix} \dfrac{\partial^2 w}{\partial x^2} \\ \dfrac{\partial^2 w}{\partial y^2} \\ \dfrac{2\partial^2 w}{\partial x \partial y} \end{bmatrix} dxdy \tag{50}$$

where $C = D \begin{bmatrix} 1 & v & 0 \\ v & 1 & 0 \\ 0 & 0 & \frac{1}{2}(1-v) \end{bmatrix}$ From Equation (48)

$$\begin{bmatrix} \dfrac{\partial^2 w}{\partial x^2} \\ \dfrac{\partial^2 w}{\partial y^2} \\ \dfrac{2\partial^2 w}{\partial x \partial y} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 2 & 0 & 0 & 6x & 2y & 0 & 0 & 6xy & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 2x & 6y & 0 & 6xy \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 4x & 4y & 0 & 6x^2 & 6y^2 \end{bmatrix} a = Ga \tag{51}$$

Using Eqs. (49) and (51) in equation (50)

$$\mathrm{d}_e = \frac{1}{2} w^T{}_e K_e w_e$$

from Eqs. (48 where the stiffness matrix

$$K_e = B^T \left[\int_0^{l_y} \int_0^{l_x} G^T C G dxdy \right] B \tag{52}$$

The kinetic energy of the element

$$\Im_e = \frac{1}{2} \int_0^{l_y} \int_0^{l_x} \mathrm{r}\, h \left(\frac{\partial w}{\partial t} \right)^2 dydx$$

) and (49) $\quad \frac{\partial w}{\partial t} = gB\dot{w}_e$

Thus

$$\Im_e = \frac{1}{2} \dot{w}_e^T M_e \dot{w}_e$$

where the element mass matrix

$$M_e = B^T \left[\mathrm{r}\, h \int_0^{l_y} \int_0^{l_x} g^T dydx \right] B \tag{53}$$

The structure matrices are assembled in the manner described in Section 5.4, noting that element matrices are now of order 12 × 12 and there are three degrees of freedom per node. Explicit expressions for matrices K_e and M_e can be found in books on finite element method.

For forced vibration the matrix equation

$$M\ddot{w} + Kw = p \tag{54}$$

This is obtained from the Lagrange equation. In equation (54) **K** and **M** are the stiffness and mass matrices of the structure, obtained by assembling element matrices after converting from local to global coordinates if necessary and after eliminating rows and columns associated with zero nodal values at boundaries. The vector **w** comprises all the degrees of freedom of the constrained structure; the vector **p** consists of the generalized forces associated with each nodal variable. Considering the contribution to **p** from a particular element, $\mathbf{p}_e$, and supposing that this element is subjected to a transverse applied force per unit area of $p(x,y)f(t)$,application of the principle of virtual work gives.

$$p^T{}_e \delta w_e = \int_0^{l_y}\int_0^{l_x} p(x,y) f(t) \delta w(x,y) dydx$$

Where $\delta \mathbf{w_e}$ list the virtual increments in the element nodal values and $\delta w(x,y)$ is the virtual displacement ay point (x,y). Using the Eqs. (48) and (49),

$$p^T{}_e \delta w_e = f(t) \int_0^{l_y}\int_0^{l_x} p(x,y) gB \delta w_e dydx$$

and

$$p^T{}_e = f(t) \left[\int_0^{l_y}\int_0^{l_x} p(x,y) g dydx \right] B \tag{55}$$

With the aid of equation (55) the generalized force vector for the structure, **p** can be assembled. After solving equation (54), the stress resultants (i.e. the bending moments M_x and M_y and the twisting moment M_{xy}, and hence the stresses, for a particular element can be determined, as the stress resultants can be expressed in terms of the vector of nodal displacements $\mathbf{w_e}$ the relation [from equations (17), (49) and (51)]

$$\begin{bmatrix} M_x \\ M_y \\ M_{xy} \end{bmatrix}_e = -DGB_e \tag{56}$$

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