Polynomials in graph theory

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Abstract

In this paper we observe the problem of counting graph colorings using polynomials. Several reformulations of The Four Color Conjecture are considered (among them algebraic, probabilistic and arithmetic). In the last section Tutte polynomials are mentioned.

1. Introduction

The problem considered is coloring countries on an island (or, which is equivalent because of homeomorphism, on a sphere). Countries are planar regions. In case of *proper coloring*, 2 neighboring countries have different colors. We call arbitrary assignment of colors, not necessarily proper, *coloring*.

Boundaries should be Jordan curves, i.e. continuous images of segments. This condition implies that there would not be infinitely many countries neighboring to each other.

If we consider countries as vertices of graph and connect neighboring countries by an edge, then we can reformulate the problem in terms of coloring the graph. Now vertices of this graph should be colored in a proper way, so that 2 adjacent vertices have different colors.

The problem can easily be reduced to the case of *3-valent graph*. It's because we can surround a vertex with valency greater than 3 with a neighborhood so that in the resulting graph each vertex gets valency of three.

The last term to be introduced here is a *line graph*: its vertices correspond to edges of initial graph, two vertices are connected by edge if and only if 2 corresponding edges of initial graph are incidental.

In 1852 Guthrie proposed *The Four Color Conjecture* (*4CC*). It claims that every map can be colored in 4 colors. There were many attempts to prove the 4CC, and in 1976 Appel and Haken gave their proof that can be called ”computer-to-computer” one. It means that the proof was produced by machine and cannot be checked by a human due to its complexity.

Mathematicians are searching for reformulations of the 4CC, that can lead to proofs which can be checked by human beings. To such reformulations our paper is devoted. And polynomials became the main instrument of counting colorings.

2. Algebraic methods of counting graph colorings

[9]

Let's denote number of proper colorings of graph G in p colors by $\chi_G(p)$.
Consider following polynomials:

$$N_p(y,z) = p - 1 - y^{p-1}z - y^{p-2}z^2 - \ldots - yz^{p-1}$$

$$M_G(p) = \prod_{(v_k,v_l)\in E} N_p(x_k, x_l)$$

Variables correspond here to objects we are going to color, i.e. vertices.

Operator $\mathcal{R}_p$ replaces the exponent of each variable x_p by its value modulo p: $x_i^{3p+1} \mapsto x_i$.

Theorem 2.1 *For any graph* $G = (V, E)$, $|V| = m$, $|E| = n \quad \forall p \in \mathbb{N}$

$$\chi_G(p) = p^{m-n} (\mathcal{R}_p M_p(G))(0, \dots, 0).$$

Proof of the theorem

Polynomial $\mathcal{R}_p M_p(G)$ can be uniquely determined by choosing appropriate p^m values of set of variables (due to interpolation process).

Let $c_0 = 1, c_1 = \omega, \dots, c_{p-1} = \omega^{p-1}$ be colors (where ω is the primitive root of 1 of degree p).

$\mu : V \to C = \{c_0, \dots, c_{p-1}\}$ is a coloring of the graph G.

Using interpolation theorem we get:

$$\mathcal{R}_p M_p(G) = \sum_{\mu} (\mathcal{R}_p M_p(G))(\mu(v_1), \dots, \mu(v_m)) P_\mu$$

(the sum is taken through all p^m colorings),

$$P_\mu = \prod_{k=1}^{m} S_p(x_k, \mu(v_k)),$$

$$S_p(x, c_q) = \prod_{0 \le l \le p-1,\, l \ne q} \frac{x - c_l}{c_q - c_l}.$$

Intuitive sense of the interpolation formula can be explained in such manner: x in

$$S_p(x, c_q) = \prod_{0 \le l \le p-1,\, l \ne q} \frac{x - c_l}{c_q - c_l}$$

takes values from $\{c_0, \dots, c_{p-1}\}$, so $S_p(x, c_q) = 1$ for $x = c_q$ and $S_p(x, c_q) = 0$ for $x = c_l,\ l \ne q$. Therefore the value of $\mathcal{R}_p M_p(G)$ on some coloring, if we look at the right-hand side of the equality, is computed correctly.

For arguments from C (the set of colors) values of $M_p(G)$ and $\mathcal{R}_p M_p(G)$ are the same, so we have

$$\mathcal{R}_p M_p(G) = \sum_{\mu} M_p(G)(\mu(v_1), \dots, \mu(v_m)) P_\mu$$

$$N_p(y, y) = p - 1 - y^p - \dots - y^p = 0,$$

$$N_p(y, z) = p - \frac{y^p - z^p}{y - z} y = p \quad \text{if } y \ne z,$$

So $M_p(G)(\mu(v_1), \dots, \mu(v_m)) = p^n$ for a proper coloring,
else $M_p(G)(\mu(v_1), \dots, \mu(v_m)) = 0$. We came to

$$\mathcal{R}_p M_p(G) = p^n \sum_{\mu} P_\mu$$

where sum is taken through $\chi_G(p)$ proper colorings.

Substituting $x_1 = \dots = x_m = 0$:

$$P_\mu(0, \dots, 0) = \prod_{k=1}^{m} S_p(0, \mu(v_k)),$$

$$S_p(0, c_q) = \prod_{0 \le l \le p-1,\, l \ne q} \frac{-c_l}{c_q - c_l} = \prod_{0 \le l \le p-1,\, l \ne q} \frac{1}{1 - \frac{c_q}{c_l}} = \prod_{1 \le l \le p-1} \frac{1}{1 - \omega^l}$$

So $S_p(0, c_q) = S_p(0)$ is independent of c_q.

$$(\mathcal{R}_p M_p(G))(0,\ldots,0) = p^n S_p(0)^m \chi_G(p)$$

If we substitute any specific graph (e.g. K_1 that has 1 vertex and 0 edges) we get $S_p(0)$:
$m = 1,\ n = 0$, $\mathcal{R}_p M_p(K_1))(0,\ldots,0) = 1$ (void product – no vertices in the graph), $\chi_{K_1}(p) = p$.
Therefore $S_p(0) = p^{-1}$ and $\chi_G(p) = p^{m-n}(\mathcal{R}_p M_p(G))(0,\ldots,0)$.

Colorings of 3-valent graphs

Let G be a planar 3-valent graph (each vertex has 3 adjacent vertices). 3-valent graph T can be represented as $T = \{\langle e_{i_1}, e_{j_1}, e_{k_1}\rangle, \ldots, \langle e_{i_{2n}}, e_{j_{2n}}, e_{k_{2n}}\rangle\}$ ($2n$ vertices, $3n$ edges; each triple represents 3 vertices that are adjacent to this one).
$\lambda_G(p)$ is the number of (proper) colorings of edges of G in p colors.

$$L(x_p, x_q, x_r) = (x_p - x_q)(x_q - x_r)(x_r - x_p)$$

$$M(G) = \prod_{l=1}^{2n} L(x_{i_l}, x_{j_l}, x_{k_l})$$

$$\mathcal{R}_3 M(G)(x_1,\ldots,x_{3n}) = \sum_{d_1,\ldots,d_{3n}\in\{0,1,2\}} c_{d_1,\ldots,d_{3n}} x_1^{d_1}\ldots x_{3n}^{d_{3n}}$$

Variables in the formulas above correspond to edges.

Theorem 2.2 *For any planar 3-valent graph G*

$$\lambda_G(3) = (\mathcal{R}_3 M(G))(0,\ldots,0) = c_{0,\ldots,0}.$$

Proof.

Here coloring is defined as $\nu : E \to \{1, \omega, \omega^2\}$ where $\omega = \frac{-1+i\sqrt{3}}{2}$, the primitive cubic root of 1.
By interpolation theorem:

$$\mathcal{R}_3 M(G) = \sum_\nu M(G)(\nu(e_1),\ldots,\nu(e_{3n}))P_\nu,$$

$$P_\nu = \prod_{k=1}^{3n} S_3(x_k, \nu(e_k)),$$

summation is taken through all 3^{3n} colorings, but really through proper ones because

$$M(G)(\nu(e_1),\ldots,\nu(e_{3n})) =$$

$$= \prod_{l=1}^{2n}(\nu(e_{i_l}) - \nu(e_{j_l}))(\nu(e_{j_l}) - \nu(e_{k_l}))(\nu(e_{k_l}) - \nu(e_{i_l}))$$

equals 0 if ν is not a proper coloring.

If the coloring ν is proper
then there are $1, \omega, \omega^2$ between $\nu(e_p), \nu(e_q), \nu(e_r)$,
so $L(\nu(e_p), \nu(e_q), \nu(e_r)) = \pm i3\sqrt{3}$,
and $M(G)(\nu(e_1),\ldots,\nu(e_{3n})) = \pm 3^{3n}$. The proper sign is $+$, as can be proven by induction on n
Then $\mathcal{R}_3 M(G) = 3^{3n}\sum_\nu P_\nu$,
here are $\lambda_G(3)$ summands, according to the number of proper colorings.
$(\mathcal{R}_3 M(G))(0,\ldots,0) = 3^{3n}(S_3(0))^{3n}\lambda_G(3) = \lambda_G(3)$ ($S_3(0) = \frac{1}{3}$ – see previous theorem).

3. Counting graph colorings in terms of orientations

[5, 1, 2]
$G = (V, E),\ f : V \to \mathbb{Z}$.
G is f**-choosable** if $\forall S : V \to 2^{\mathbb{Z}},\ |S(v)| = f(v)\,\forall v$
there exists a proper coloring $c : V \to \mathbb{Z}$ such that $\forall v\ c(v) \in S(v)$.

G is k**-choosable** ($k \in \mathbb{Z}$) if $f \equiv k$ – i.e., for every vertex for any set of k potential colors there exists a proper coloring. Minimal k for which G is k-choosable is referred to as a **choice number** of G.

Let $\chi(G), \chi'(G)$ be chromatic numbers of G and line graph of G resp., and ch(G), ch′(G) choice numbers.
Obviously for any graph G holds ch(G) $\geq \chi(G)$. If G is k-choosable then it is k-colorable: it's enough to take $S = \{1, \ldots, k\}\ \ \forall v$.
There exist graphs with ch(G) $> \chi(G)$ – see the figure where $\chi(G) = 2$, but G is not 2-choosable. But there is a conjecture

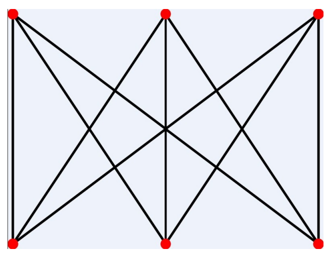

Figure 1. $S(u_i) = S(v_i) = \{1, 2, 3\} \setminus \{i\}$

claiming that $\forall G$ ch′(G) $= \chi'(G)$.
We consider oriented graphs (digraphs) and introduce some concepts.
Eulerian graph: for each vertex its indegree equals its outdegree.
Even (**odd**) graph: a graph with even (odd) number of edges.
$EE(D)$: a number of even Eulerian subgraphs of graph D.
$EO(D)$: a number of odd Eulerian subgraphs of graph D.
$d^+_D(v)$: outdegree of vertex v in D.

Theorem 3.1 *$D = (V, E)$ being a digraph, $|V| = n,\ d_i = d^+_D(v_i)$,*
$f(i) = d_i + 1\ \forall i \in \{1, \ldots, n\}, \quad EE(D) \neq EO(D) \Rightarrow$
D is f-choosable.

Proof of the theorem

Lemma 3.1 *Let $P(x_1, \ldots, x_n)$ be polynomial in n variables over $\mathbb{Z}$, for $1 \leq i \leq n$ the degree of P in x_i doesn't exceed d_i,*
$S_i \subset \mathbb{Z} :\ |S_i| = d_i + 1$.
If $\forall (x_1, \ldots, x_n) \in S_1 \times \ldots \times S_n\ P(x_1, \ldots, x_n) = 0$ then $P \equiv 0$.

(Proof by induction.)
Graph polynomial of undirected graph G:

$$f_G(x_1, \ldots, x_n) = \prod_{i<j, v_i v_j \in E} (x_i - x_j)$$

Monomials of that polynomial are in natural correspondence with orientations of G (recall that orientation is a choice of the direction of an edge, and here we choose one variable from each pair of brackets).
We call edge $v_i v_j$ **decreasing** if $i > j$. Orientation is **even** if it has even number of decreasing edges, else it's **odd**.
$DE(d_1, \ldots, d_n)$ and $DO(d_1, \ldots, d_n)$ are sets of even and odd orientations;
here non-negative numbers d_i correspond to outdegrees of vertices. Then evidently holds

Lemma 3.2

$$f_G(x_1,\ldots,x_n) = \sum_{d_1,\ldots,d_n\geq 0} (|DE(d_1,\ldots,d_n)| - |DO(d_1,\ldots,d_n)|) \prod_{i=1}^{n} x_i^{d_i}$$

Let us further take $D_1, D_2 \in DE(d_1,\ldots,d_n) \cup DO(d_1,\ldots,d_n)$.
$D_1 \oplus D_2$ denotes set of edges in D_1 that have the opposite direction in D_2.

Mapping $D_2 \mapsto D_1 \oplus D_2$ is a bijection between $DE(d_1,\ldots,d_n) \cup DO(d_1,\ldots,d_n)$ and set of Eulerian subgraphs of D_1.

If D_1 is even then it maps even orientations to even subgraphs and odd ones to odd ones.
If D_1 is odd then it maps even to odd and odd to even.

Thus we get

$$||DE(d_1,\ldots,d_n)| - |DO(d_1,\ldots,d_n)|| = |EE(D_1) - EO(D_1)|$$

(it's the coefficient of the monomial $\prod x_i^{d_i}$ in f_G).

Recall the statement of the theorem. Suppose there is no such coloring.

Then $\forall (x_1,\ldots,x_n) \in S_1 \times \ldots \times S_n$ $f_G(x_1,\ldots,x_n) = 0$, where $S_i = \{1,\ldots,d_i+1\}$.

Let $Q_i(x_i)$ be

$$Q_i(x_i) = \prod_{s\in S_i} (x_i - s) = x_i^{d_i+1} - \sum_{j=0}^{d_i} q_{ij} x_i^j.$$

If $x_i \in S_i$ then $x_i^{d_i+1} = \sum_{j=0}^{d_i} q_{ij} x_i^j$.
We are going to replace in f_G each occurrence of $x_i^{f_i}$, $f_i > d_i$, by a linear combination of smaller powers (using the above equality). So we get polynomial $\tilde{f_G}$.
$\forall (x_1,\ldots,x_n) \in S_1 \times \ldots \times S_n$ $\tilde{f_G}(x_1,\ldots,x_n) = 0$ and by first Lemma $\tilde{f_G} \equiv 0$. But coefficient of $\prod_{i=1}^n x_i^{d_i}$ in f_G is nonzero, and it remains the same in $\tilde{f_G}$ due to homogeneity of f_G. We come to a contradiction. The proof is done.

4. Probabilistic restatement of Four Color Conjecture

[8, 7]

G is a 3-valent biconnected undirected graph with $2n$ vertices, $3n$ edges.
Its undirected line graph F_G has $3n$ vertices (each of degree 4) and $6n$ edges. We assign the same probability to each of 2^{6n} orientations that can be attached to F_G.

Consider 2 orientations F'_G and F''_G.

Orientations have the same **parity** if they differ in direction of even number of edges.

Orientations F'_G, F''_G are **equivalent modulo 3** if for every vertex its outdegree in F'_G equals modulo 3 its outdegree in F''_G.

Event A_G: 2 randomly chosen orientations have the same parity.
Event B_G: 2 randomly chosen orientations are equivalent modulo 3.

Theorem 4.1 *For any biconnected planar 3-valent graph G having $2n$ vertices*

$$P\left(B_G|A_G\right) - P(B_G) = \left(\frac{27}{4096}\right)^n \cdot \frac{\chi_G(4)}{4}.$$

So 4CC is equivalent to the statement
that there is a positive correlation between A_G and B_G.

It can be proved by considering 2 graph polynomials:
$M' = \prod_{e_ie_j\in L_G}(x_i - x_j)$ (here L_G is a set of edges of F_G),
$M'' = \prod_{e_ie_j\in L_G}(x_i^2 - x_j^2)$.

5. Arithmetical restatement of Four Color Conjecture

[6]
The Main Theorem

Theorem 5.1 $\exists p, q \in \mathbb{N},\ \ 4q$ *linear functions* $A_k(m, c_1, \ldots, c_p), B_k(m, c_1, \ldots, c_p), C_k(m, c_1, \ldots, c_p), D_k(m, c_1, \ldots, c_p)$, $k \in \{1, \ldots, q\}$ *such that 4CC is equivalent to the following statement:*

$$\forall m, n\ \exists c_1, \ldots, c_p\ E(n, m, c_1, \ldots, c_p) \not\equiv 0 \mod 7,$$

$$E(n, m, c_1, \ldots, c_p) = \begin{pmatrix} A_k(m, c_1, \ldots, c_p) + 7^n B_k(m, c_1, \ldots, c_p) \\ C_k(m, c_1, \ldots, c_p) + 7^n D_k(m, c_1, \ldots, c_p) \end{pmatrix}.$$

The complex representation of E's in the form of binomial coefficients comes from computation and has nothing to do with a practical value of the Main Theorem. It just claims that, having $E(n, m, c_1, \ldots, c_p)$, we can take $G(m, n)$ whose values are never divisible by 7,
arbitrary integer-valued functions $F(n, m, c_1, \ldots, c_p)$, and

$$\sum_{c_1} \ldots \sum_{c_p} E(n, m, c_1, \ldots, c_p) F(n, m, c_1, \ldots, c_p) = G(m, n)$$

implies the 4CC.

We are to come to the main theorem from original 4CC step by step using reformulations.

- Firstly, we will color not countries but their capitals (2 capitals are connected by road iff countries are neighbors).
- Then we introduce **internal** and **external** (as a whole – **ranked**) edges ($G = \langle V, E_I, E_X \rangle$). Ends of internal edge should have the same color, ends of external edge should be colored differently. Now the 4CC sounds as follows:
  If a planar graph with ranked edges can be colored in some number of colors (more precisely in 6 colors – it's always possible) then it can be colored in 4 colors.
- Then we say: we have a graph with vertices (V) colored somehow and edges (E).
  2 colorings are **equivalent** if they induce the same division of E in 2 groups (internal and external).
  For every coloring we are searching for the equivalent one consisting of 4 colors.
- The next term to introduce is **spiral graph**.
  It has infinitely many vertices k.
  2 vertices i, j are connected by edge iff $|i - j| = 1$ or $|i - j| = n$.
  We color this construct in colors from $\{0, \ldots, 6\}$ so that finitely many vertices have color greater than 0. 4-coloring λ for a given coloring μ is made paying attention to 3 properties:

  1. $\lambda(k) = 0 \iff \mu(k) = 0$
  2. $\lambda(k) = \lambda(k+1) \iff \mu(k) = \mu(k+1)$
  3. $\lambda(k) = \lambda(k+n) \iff \mu(k) = \mu(k+n)$

  On the picture you can see an example of spiral graph, with $n = 8$.

  The structure is introduced in order to simplify the presentation of a graph. Now we can consider only graphs with relatively regular structure (such as spiral) instead of arbitrary planar graph induced by countries, capitals and roads.
- We can represent our colorings as a natural number in **base-7 notation**: $m = \sum_{k=0}^{\infty} \mu(k) 7^k,\ l = \sum_{k=0}^{\infty} \lambda(k) 7^k$.
  Our requirements to λ imply that:

  1. there are no 7-digits '5' and '6' in l
  2. the k-th digit of l equals 0 $\iff$ the k-th digit of m equals 0
  3. the $(k+1)$-th digit equals the k-th digit at the same time for both l and m

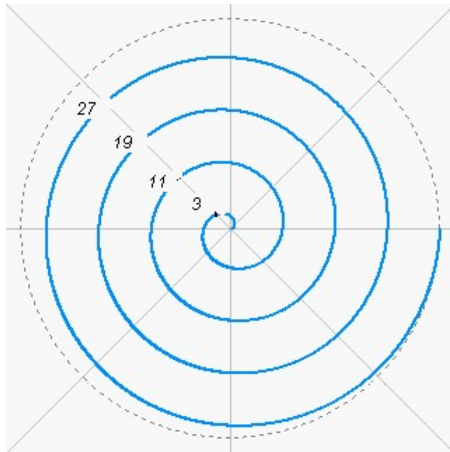

Figure 2. Spiral graph

4. the $(k+n)$-th digit equals the k-th digit at the same time for both l and m

- We can view m as $\sum_{i=1}^{6} im_i$ so that $m_i = \sum_{\mu(k)=i} 7^k$.
  Now we need 2 more definitions.
  $b \in \mathbb{Z}_+$ is **Bool** if its 7-digits are either 0 or 1.
  Boolean numbers a and b are said to be **orthogonal** ($a \perp b$) if they never have '1' in the same position.
  We introduce $c_{ij} = \sum_{\mu(k)=i, \lambda(k)=j} 7^k$
  and know that Bool(c_{ij}), $c_{ij} \perp c_{i'j'}$ if $\langle i, j \rangle \neq \langle i', j' \rangle$.
  $m_i = \sum_{j=1}^{6} c_{ij}, \; l_j = \sum_{i=1}^{4} c_{ij}$
  Now conditions on 4-coloring are following (not counting those we've already seen):

  1. $7c_{ij} \perp c_{ij'}, \; j \neq j'$
  2. $7c_{ij} \perp c_{i'j}, \; i \neq i'$
  3. $7^n c_{ij} \perp c_{ij'}, \; j \neq j'$
  4. $7^n c_{ij} \perp c_{i'j}, \; i \neq i'$

To perform the last step, we need some algebraic facts.

Theorem 5.2 ***(E. E. Kummer)***
A prime number p comes into the factorization of the binomial coefficient

$$\binom{a+b}{a}$$

with the exponent equal to the number of carries performed while computing sum $a+b$ in base-p positional notation.

There are several corollaries that simply follow the theorem:
The first corollary:
All 7-digits of a are less or equal to 3 if and only if

$$\binom{2a}{a} \not\equiv 0 \mod 7.$$

The second corollary:

$$\text{Bool}(a) \iff \binom{2a}{a}\binom{4a}{2a} \not\equiv 0 \mod 7.$$

The third corollary:

$$\mathrm{Bool}(a)\&\mathrm{Bool}(b) \Rightarrow$$
$$\left[a\perp b \iff \begin{pmatrix} 2(a+b) \\ a+b \end{pmatrix} \begin{pmatrix} 4(a+b) \\ 2(a+b) \end{pmatrix} \not\equiv 0 \mod 7 \right].$$

The last corollary:

$$\mathrm{Bool}(a)\&\mathrm{Bool}(b) \Rightarrow \left[a\perp b \iff \begin{pmatrix} 4(a+b) \\ 2(a+b) \end{pmatrix} \not\equiv 0 \mod 7 \right].$$

The last reformulation is implied by corollaries and leads to the Main Theorem.

The last reformulation:
$\forall n, m \in \mathbb{Z}_+ \; \exists c_{ij} \in \mathbb{Z}_+, \; i \in \{1, \ldots, 6\}, j \in \{1, \ldots, 4\}$:
none of 986 binomial coefficients is divisible by 7:

$$\begin{pmatrix} 2c_{ij} \\ c_{ij} \end{pmatrix}, \begin{pmatrix} 4c_{ij} \\ 2c_{ij} \end{pmatrix}, \begin{pmatrix} 4(c_{i'j'} + c_{i''j''}) \\ 2(c_{i'j'} + c_{i''j''}) \end{pmatrix}, \langle i', j' \rangle \neq \langle i'', j'' \rangle,$$

$$\begin{pmatrix} 4(7c_{ij} + c_{ij'}) \\ 2(7c_{ij} + c_{ij'}) \end{pmatrix}, j \neq j', \begin{pmatrix} 4(7c_{ij} + c_{i'j}) \\ 2(7c_{ij} + c_{i'j}) \end{pmatrix}, i \neq i',$$

$$\begin{pmatrix} 4(7^n c_{ij} + c_{ij'}) \\ 2(7^n c_{ij} + c_{ij'}) \end{pmatrix}, j \neq j', \begin{pmatrix} 4(7^n c_{ij} + c_{i'j}) \\ 2(7^n c_{ij} + c_{i'j}) \end{pmatrix}, i \neq i',$$

$$\begin{pmatrix} m \\ C \end{pmatrix}, \begin{pmatrix} C \\ m \end{pmatrix},$$

$$\text{where } C = \sum_{i=1}^{6} \left(\sum_{j=1}^{4} c_{ij} \right).$$

6. The Tutte polynomial

[3, 4]

Let $G = (V, E)$ be a (multi)graph (it may have loops and multiple edges). On such a graph one can perform following operations:

1. *cut* : $G - e$, where $e \in E$ (delete the edge e)
2. *fuse* : $G \setminus e$, where $e \in E$ (delete the edge e and join vertices incident to e)

If $k(G)$ is a number of components in $G = (V, E)$ then *rank* of graph G is $r(G) = |V| - k(G)$, and *nullity* of graph G is $n(G) = |E| - |V| + k(G)$.

For any subgraph F we write $k\langle F\rangle, r\langle F\rangle, n\langle F\rangle$. Then

$$S(G; x, y) = \sum_{F \subset E(G)} x^{r\langle E\rangle - r\langle F\rangle} y^{n\langle F\rangle} = \sum_{F \subset E(G)} x^{k\langle F\rangle - k\langle E\rangle} y^{n\langle F\rangle}$$

is called *rank-generating polynomial*.

Theorem 6.1

$$S(G; x, y) = \begin{cases} (x+1)S(G-e; x, y), & \text{e is a bridge,} \\ (y+1)S(G-e; x, y), & \text{e is a loop,} \\ S(G-e; x, y) + S(G \setminus e; x, y), & \text{otherwise.} \end{cases}$$

Furthermore, $S(E_n; x, y) = 1$ *for an empty graph* E_n *with* n *vertices.*

This can be easily proved if we form two groups of F's (subsets of $E(G)$): those which include e (the edge to be eliminated) and those which do not — and investigate simple properties of rank and nullity.

The *Tutte polynomial* is defined as follows:

$$T_G(x,y) = S(G; x-1, y-1) = \sum_{F \subset E(G)} (x-1)^{r\langle E\rangle - r\langle F\rangle}(y-1)^{n\langle F\rangle}$$

Of course the appropriate statement holds:

$$T_G(x,y) = \begin{cases} xT_{G-e}(x,y), & \text{e is a bridge,} \\ yT_{G-e}(x,y), & \text{e is a loop,} \\ T_{G-e}(x,y) + T_{G\backslash e}(x,y), & \text{otherwise.} \end{cases}$$

Theorem 6.2 *There is a unique map U from the set of multigraphs to the ring of polynomials over $\mathbb{Z}$ of variables $x, y, \alpha, \sigma, \tau$ such that*
$U(E_n) = U(E_n; x, y, \alpha, \sigma, \tau) = \alpha^n$ *and*

$$U(G) = \begin{cases} xU_{G-e}(x,y), & \text{e is a bridge,} \\ yU_{G-e}(x,y), & \text{e is a loop,} \\ \sigma U_{G-e}(x,y) + \tau U_{G\backslash e}(x,y), & \text{otherwise.} \end{cases}$$

Furthermore,

$$U(G) = \alpha^{k(G)}\sigma^{n(G)}\tau^{r(G)}T_G(\alpha x/\tau, y/\sigma).$$

Proof sketch:
$U(G)$ is a polynomial of σ and τ because $deg_x T_G(x,y) = r(G)$, $deg_y T_G(x,y) = n(G)$. The uniqueness of U is implied by constructive definition. It can be checked simply that $U(E_n) = \alpha^n$ and recurrent equalities hold for $U(G) = \alpha^{k(G)}\sigma^{n(G)}\tau^{r(G)}T_G(\alpha x/\tau, y/\sigma)$.

Definition 6.1 *If $p_G(x)$ is the number of proper colorings of vertices of graph G in x colors then $p_G(x)$ is called the* ***chromatic function*** *of G.*

Corollary 6.1

$$p_G(x) = (-1)^{r(G)}x^{k(G)}T_G(1-x, 0)$$

So the chromatic function is actually the *chromatic polynomial*.
Proof sketch:

$$p_{E_n}(x) = x^n$$

and $\forall e \in E(G)$

$$p_G(x) = \begin{cases} \frac{x-1}{x}p_{G-e}(x), & \text{e is a bridge,} \\ 0, & \text{e is a loop,} \\ p_{G-e}(x) - p_{G\backslash e}(x), & \text{otherwise.} \end{cases}$$

Therefore

$$p_G(x) = U(G; \frac{x-1}{x}, 0, x, 1, -1) = x^{k(G)}(-1)^{r(G)}T_G(1-x, 0).$$

With the aid of the Tutte polynomial, one can calculate a variety of number characteristics concerning graphs, among them number of spanning trees.

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