Integer Relations among Real Numbers

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Outline

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$1/0.1412742382271468144 \approx 7.0784313725490196080$

$1/0.0784313725490196078431 39 \approx 12.7499999999999999975$

$1/0.7499999999999999975 \approx 1.3333333333333333778$

$1/0.3333333333333333778 \approx 2.9999999999999995998$

$1/0.9999999999999995998 \approx 1.0000000000000004002$

$1/0.0000000000000004002 \approx 2498750624687656 1,719$

Continuous Fractions

$$0.14127423822714681440\ldots \approx \cfrac{1}{7+\cfrac{1}{12+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{24987506246876561,719}}}}}} \Rightarrow$$

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(Actually the period of $\dfrac{51}{361}$ is 342.)

Quadratic Irrationalities

Now suppose we don't know if X is rational. But we do know that it is a *quadratic irrationality*. (That is a root of an equation $ax^2 + bx + c = 0$ with a, b, c rational.)

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Theorem(Lagrange)

X is a quadratic irrationality $\Leftrightarrow$ its continuos fraction is periodic.

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α is an *algebraic number* if there exist $a_0, \ldots, a_n \in \mathbb{Z}$ such that $a_n\alpha^n + \ldots + a_1\alpha + a_0 = 0$ and $a_n \neq 0$. The *degree* of α is the smallest of such n.

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Definition

An *integer relation* for n-tuple $(x_1, \ldots, x_n) \in \mathbb{R}^n$ is an n-tuple $0 \neq (a_1, \ldots, a_n) \in \mathbb{Z}^n$ such that $a_1x_1 + \ldots + a_nx_n = 0$.

Algorithms for Finding Integral Relations

The problem of finding an integer relation for two numbers (x_1, x_2) can be solved by applying the Euclidian algorithm to x_1, x_2, or, equivalently, by computing the continued fraction expansion of x_1/x_2.

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The best known and most used algorithms at the present time are either algorithms based on lattice basis reduction algorithm by Lenstra, Lenstra, Jr. and Lovász (**LLL**) or **PSLQ algorithm** based on ideas of Ferguson, Forcade and Bergman. (Both discovered in 1970-s –1980-s.)

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Some Reminders from Linear Algebra

- Let $\mathbb{R}^n$ be the n -dimensional real vector space ($n > 1$) with inner product: $\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{j=1}^{n} x_j y_j$.

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- $\mathbf{x}$ and $\mathbf{y}$ are *orthogonal* $\Leftrightarrow \langle \mathbf{x}, \mathbf{y} \rangle = 0$.

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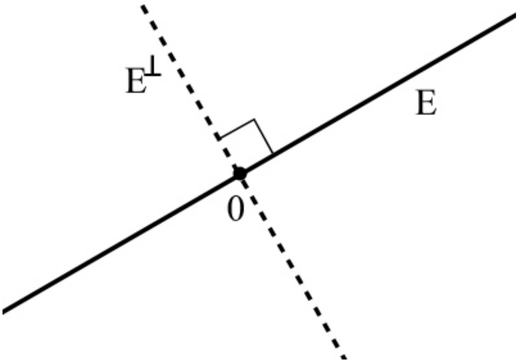
- For a linear subspace $E \subset \mathbb{R}^n$ we denote by $E^\perp \subset \mathbb{R}^n$ the *orthogonal complement* of E (i.e., the subspace consisting of all vectors that are orthogonal to E).

Some Reminders from Linear Algebra

E and $E^{\perp}$:



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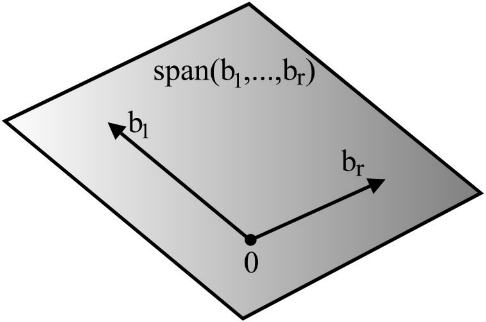
- $\mathrm{span}(\mathbf{b}_1, \ldots, \mathbf{b}_r)$ is the linear space, spanned on $\mathbf{b}_1, \ldots, \mathbf{b}_r$:
$$\mathrm{span}(\mathbf{b}_1, \ldots, \mathbf{b}_r) = \left\{ \sum_{j=1}^{j=r} a_j \mathbf{b}_j \,\middle|\, a_j \in \mathbb{R} \right\}.$$



Gram-Schmidt orthogonalization

With $\mathbf{b}_0 = \mathbf{x},\ \mathbf{b}_1, \ldots, \mathbf{b}_n \in \mathbb{R}^n$ we associate the orthogonal system $\mathbf{b}_0^*, \ldots, \mathbf{b}_n^*$ that are defined inductively:

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- $\mathbf{b}_0^* = \mathbf{x}$,
- $\mathbf{b}_i^* = \mathbf{b}_i - \sum_{j=0}^{i-1} \frac{\langle \mathbf{b}_i, \mathbf{b}_j^* \rangle}{\langle \mathbf{b}_j^*, \mathbf{b}_j^* \rangle} \mathbf{b}_j^*,\ i = 1, \ldots, n.$

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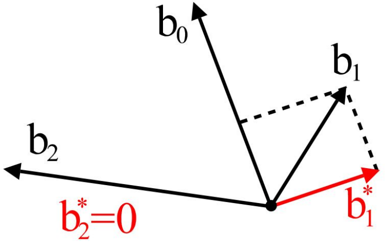

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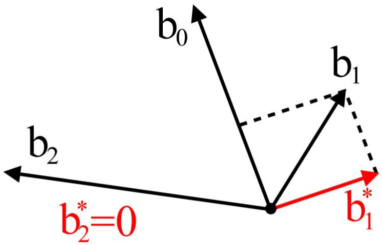

Note: $\mathbf{b}_i^*$ is orthogonal to $\mathrm{span}(\mathbf{b}_0^*, \ldots, \mathbf{b}_{i-1}^*) = \mathrm{span}(\mathbf{b}_0, \ldots, \mathbf{b}_{i-1})$.

Lattices

Definition

A *lattice* $L \subset \mathbb{R}^n$ is an additive closure of some linear independent $\mathbf{b}_1, \ldots, \mathbf{b}_r \in \mathbb{R}^n$, i.e. $L = \left\{ \sum_{i=1}^{r} m_i \mathbf{b}_i \,|\, m_i \in \mathbb{Z} \right\}$.

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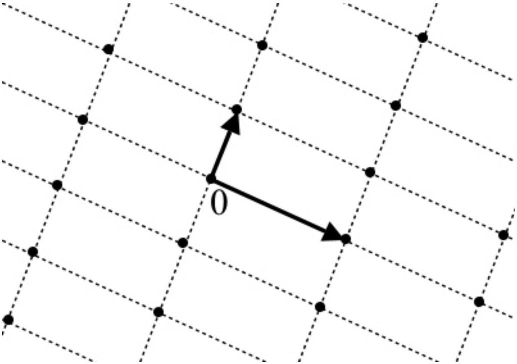

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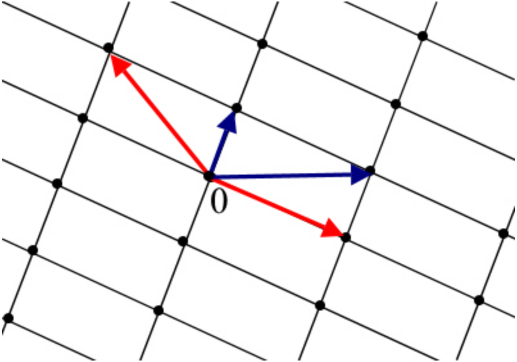

Lattices

An important example: the lattice $L_{\mathbf{x}} \subset \mathbb{Z}^n$ of all integer relations for $\mathbf{x}$ together with $\mathbf{0}$: $L_{\mathbf{x}} := \{\mathbf{m} \in \mathbb{Z}^n \,|\, \langle \mathbf{x}, \mathbf{m} \rangle = 0\}$.

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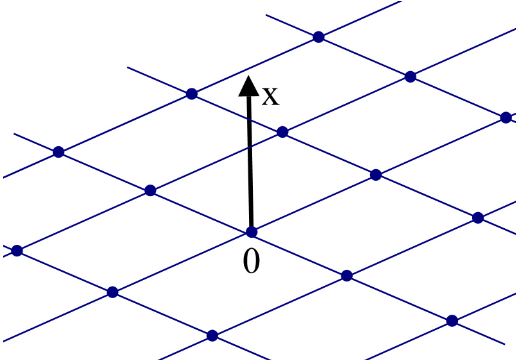

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With every basis $\mathbf{b}_1, \ldots, \mathbf{b}_n$ there is the *dual basis* $\mathbf{c}_1, \ldots, \mathbf{c}_n$:

$$[\mathbf{c}_1, \ldots, \mathbf{c}_n]^T = [\mathbf{b}_1, \ldots, \mathbf{b}_n]^{-1} \Leftrightarrow [\mathbf{c}_1, \ldots, \mathbf{c}_n]^T [\mathbf{b}_1, \ldots, \mathbf{b}_n] = \mathrm{Id}$$

$$\Leftrightarrow \langle \mathbf{b}_j, \mathbf{c}_k \rangle = \delta_{jk} = \begin{cases} 0, & j \neq k \\ 1, & j = k \end{cases} .$$

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$\Leftrightarrow \langle \mathbf{b}_j, \mathbf{c}_k \rangle = \delta_{jk} = \begin{cases} 0, & j \neq k \\ 1, & j = k \end{cases}.$

Note: $\mathbf{b}_1, \ldots, \mathbf{b}_n \in \mathbb{Z}^n$ and
$B = [\mathbf{b}_1, \ldots, \mathbf{b}_n]$ unimodular ($\det B = \pm 1$) $\Rightarrow \mathbf{c}_1, \ldots, \mathbf{c}_n \in \mathbb{Z}^n$.

HJLS: Source; Model of Computation; Notation

Source: J. Hastad, B. Just, J. C. Lagarias, and C. P. Schnorr. Polynomial Time Algorithms for Finding Integer Relations among Real Numbers.
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- Computation with real numbers.
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- $\lambda(\mathbf{x}) :=$ the length of the shortest integer relation for $\mathbf{x}$. If there are no relations then $\lambda(\mathbf{x}) := \infty$.

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(1) <u>Initiation</u>: $\mathbf{b}_0 := \mathbf{x}$; $\mathbf{b}_1, \ldots, \mathbf{b}_n :=$ standard basis of $\mathbb{Z}^n$.
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If $\|\mathbf{b}_n^*\| \neq 0$ then an integer relation is found.
Compute $[\mathbf{c}_1, \ldots, \mathbf{c}_n]^T = [\mathbf{b}_1, \ldots, \mathbf{b}_n]^{-1}$ and output the integer relation $\mathbf{c}_n$. Stop.
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Update $\mu_{i+1,j}$ for $j = 1, \ldots, i$.
Exchange $\mathbf{b}_i$ and $\mathbf{b}_{i+1}$.
Update $\|\mathbf{b}_\nu^*\|^2, \mu_{\nu j}, \; \mu_{j\nu}$ for $\nu = i, i+1, \; 1 \leq j \leq n$. Go to *(2)*.

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Note: The matrix $[\mathbf{c}_1, \ldots, \mathbf{c}_n]$ can be computed incrementally:

- Initially $[\mathbf{c}_1, \ldots, \mathbf{c}_n] = \mathrm{Id}_n$.
- $\mathbf{b}_{i+1} := \mathbf{b}_{i+1} - \lceil \mu_{i+1,i} \rfloor \mathbf{b}_i \Rightarrow \mathbf{c}_i := \mathbf{c}_i + \lceil \mu_{i+1,i} \rfloor \mathbf{c}_{i+1}$.
- $\mathbf{b}_i \leftrightarrow \mathbf{b}_{i+1} \Rightarrow \mathbf{c}_i \leftrightarrow \mathbf{c}_{i+1}$.

HJLS: Correctedness and Polynomial Time

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  Since $\langle \mathbf{b}_j, \mathbf{c}_k \rangle = 0 \ \forall k > j$ we have $\langle \mathbf{x}, \mathbf{c}_k \rangle = 0 \ \forall k > i$, in particular $\langle \mathbf{x}, \mathbf{c}_n \rangle = 0$. $\square$

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  So $\|\mathbf{m}\| \geq \dfrac{1}{\|\mathbf{b}_i^*\|}$. $\square$

Outline

PSLQ: Source; Model of Computation

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Model of Computation:

- Computation with real numbers.
- Operations: addition, subtraction, multiplication, division, comparison, the nearest integer($\lceil \rfloor$) — at unit cost.

Definitions

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Let $H_{\mathbf{x}} = (h_{i,j})$ be $n \times (n-1)$ lower-trapezoidal matrix defined by:

$$h_{i,j} := \begin{cases} 0 & 1 \leq i < j \leq n-1 \\ s_{i+1}/s_i & 1 \leq i = j \leq n-1 \\ -x_j^2/(s_j s_{j+1}) & 1 \leq j < i \leq n-1. \end{cases}$$

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Reduce H:

for $i := 2$ **to** n
    for $j := i - 1$ **to** 1 **step** -1
        $t := \lceil h_{ij}/h_{jj} \rfloor$
        $y_j := y_j + t y_i$
        for $k := 1$ **to** j
            $h_{ik} := h_{ik} - t h_{jk}$
        endfor
        for $k := 1$ **to** n
            $b_{kj} := b_{kj} + t b_{ki}$
        endfor
    endfor
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$\delta := \sqrt{h_{rr}^2 + h_{r,r+1}^2};\ \alpha := h_{rr}/\delta;\ \beta := h_{r,r+1}/\delta.$

if $r \leq n-2$ **then**

  for $i := r$ **to** n

    $h_0 := h_{ir};\ h_1 := h_{i,r+1};$

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(6)Termination: Goto *(2)* unless $y_j = 0$ for some $1 \leq j \leq n$ or $h_{ii} = 0$ for some $1 \leq i \leq n-1$.

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PSLQ is more stable, because it uses a stable matrix reduction procedure. Unfortunately, HJLS is not stable.

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PSLQ with $\gamma = \sqrt{2}$ for successive iterations $N = 0, 1, 2, 3, 4$ yields the five matrices:

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$$\begin{pmatrix} 3 & -2 & 0 \\ 1 & 2 & 1 \\ -2 & -1 & -1 \end{pmatrix}, \begin{pmatrix} \boxed{\begin{matrix} -1 \\ 5 \\ -4 \end{matrix}} & \boxed{\begin{matrix} -8 \\ 9 \\ -5 \end{matrix}} & \begin{matrix} -2 \\ 2 \\ -1 \end{matrix} \end{pmatrix}.$$

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It found 2 relations:
$-11 + 5 \cdot 27 - 4 \cdot 31 = -11 + 135 - 124 = 0;$
$-8 \cdot 11 + 9 \cdot 27 - 5 \cdot 31 = -88 + 243 - 155 = 0.$

An Example

HJLS for successive iterations $N = 0, 1, 2, 3, 4, 5, 6$ yields the seven matrices:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & -1 \end{pmatrix},$$

$$\begin{pmatrix} 1 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & -1 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & -2 \\ 1 & 3 & 2 \\ -1 & -3 & -1 \end{pmatrix},$$

$$\left(\begin{array}{ccc} 0 & -2 & \boxed{\begin{array}{c} -1 \\ 5 \\ -4 \end{array}} \\ 1 & 2 & \\ -1 & -1 & \end{array}\right).$$

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$$\begin{pmatrix} 0 & -2 & \boxed{\begin{matrix} -1 \\ 5 \\ -4 \end{matrix}} \\ 1 & 2 & \\ -1 & -1 & \end{pmatrix}.$$

It found 1 relation.

Outline

"BBP" formula for π

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Perhaps one of the best known applications of PSLQ is the 1995 discovery, by means of PSLQ computation, of the "BBP" (Bailey, Borwein, Plouffe) formula for π:

$$\pi = \sum_{k=0}^{\infty} \frac{1}{16^k} \left(\frac{4}{8k+1} - \frac{2}{8k+4} - \frac{1}{8k+5} - \frac{1}{8k+6} \right).$$

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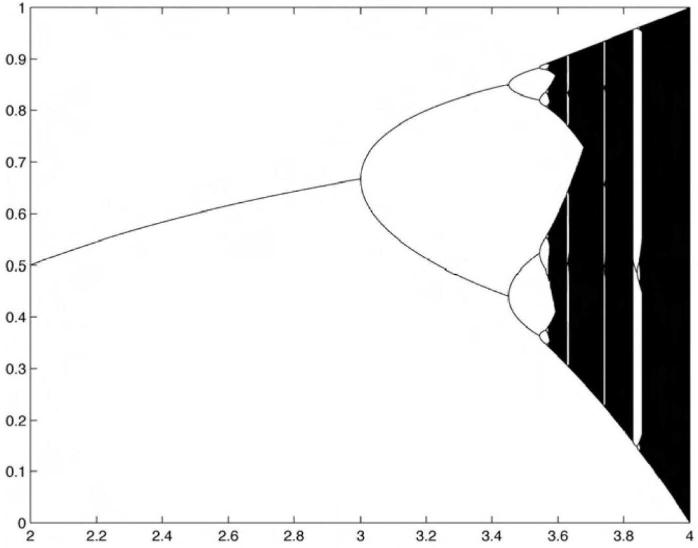
This formula permits one to compute directly hexademical digits of π without computing previous ones.

The formula was found by applying PSLQ to $(X_1, \ldots, X_n, \pi)$ where

$$X_j = \sum_{k=0}^{\infty} \frac{1}{16^k(8k+j)}.$$

Bifurcation Points in Chaos Theory

The chaotic iteration $x_{n+1} = rx_n(1 - x_n)$ ("logistic iteration"):



Bifurcation Points in Chaos Theory

$1 < r < B_1 = 3$: one limit point.
$B_1 < r < B_2 = 1 + \sqrt{6} = 3.449489\ldots$: two distinct limit points.
$B_2 < r < B_3$: four distinct limit points.
$B_3 < r < B_4$: eight distinct limit points.
And so on.

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Using PSLQ with $n = 13$ we get that B_3 satisfies:
$r^{12} - 12r^{11} + 48r^{10} - 40r^9 - 193r^8 + 392r^7 + 44r^6 + 8r^5 - 977r^4 - 604r^3 + 2108r^2 + 4913 = 0.$

Bifurcation Points in Chaos Theory

The much more difficult problem for finding B_4 was studied in

D. H. Bailey and D. J. Broadhurst. Parallel integer relation detection: techniques and applications. Mathematics of Computation, Vol.70, 2000, pp.1719-1736.

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It was conjectured that B_4 might satisfy a 240-degree polynomial, and, in addition, $\alpha = -B_4(B_4 - 2)$ might satisfy a 120-degree polynomial.
Then an advanced PSLQ implementation was employed, and a relation with coefficients descending from 257^{30} to 1 was found.

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4 year later the result was confirmed in large symbolic computation in

I. Kotsireas and K. Karamanos. Exact computation of the bifurcation point b4 of the logistic map and the Bailey-Broadhurst conjectures. Internat. J. Bifurcation and Chaos, Vol.14, 2004, pp.2417-2423.

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Further Reading

A. K. Lenstra, H. W. Lenstra, Jr., and L. Lovász.
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