Lowest Common Ancestor(LCA)
a.k.a
Nearest Common Ancestor(NCA)

Fayssal El Moufatich

Technische Universität München
St. Petersburg

JASS 2008

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- Applications
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Introduction

- One of the most fundamental algorithmic problems on trees is how to find the **Least Common Ancestor** of a pair of nodes.
- Studied intensively because:
  - It is inherently algorithmically beautifull.
  - Fast algorithms for the LCA problem can be used to solve other algorithmic problems.

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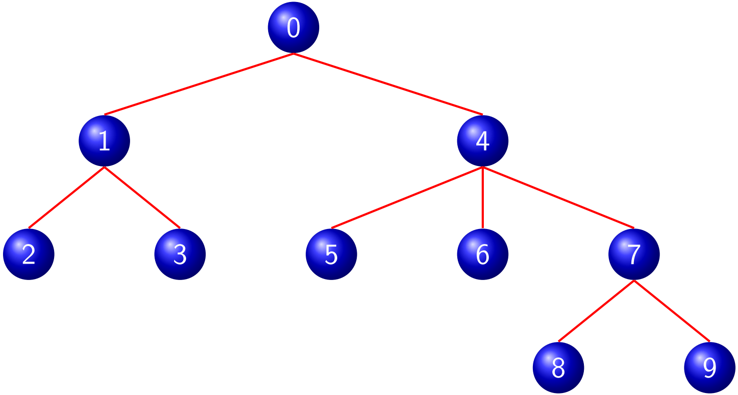

Applications

- A procedure solving the NCA problem is used by algorithms for:
  - Finding the *maximum weighted matching* in a graph.
  - Finding a *minimum spanning tree* in a graph.
  - Finding a *dominator tree* in a graph in a *directed flow-graph*.
  - Several string algorithms.
  - *Dynamic planarity testing*.
  - In *network routing*.
  - Solving various geometric problems including *range searching*.
  - Finding *evolutionary trees*.
  - And in *bounded tree-width* algorithms.
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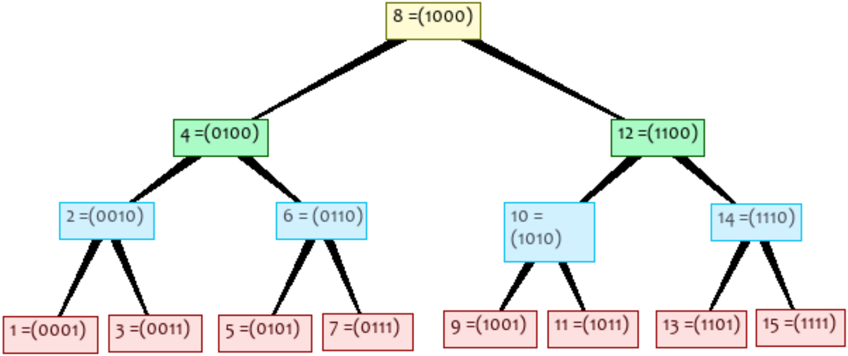

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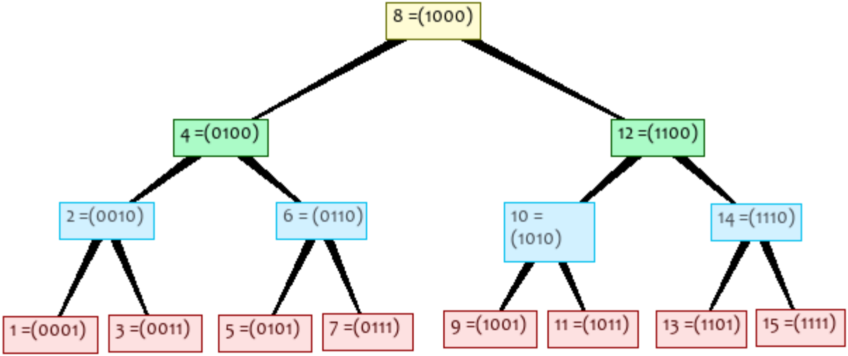

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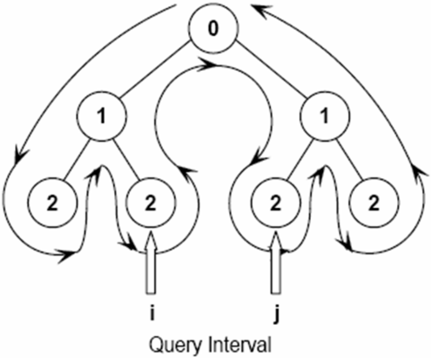

Euler tour of the tree on the left

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i j

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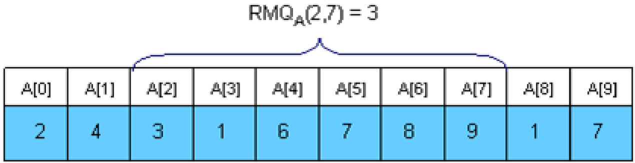
As with the DSR algorithm, LCA can be reduced to an RMQ problem.[Bender-Farach, 2000]



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- Parallel and distributed versions for algorithm exist!

Thank you for your attention!