Guaranteed Stable Projection-Based Model Reduction for Indefinite and Unstable Linear Systems *)

A paper-based study

by

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1 Introduction

Our physical systems are always modeled for analysis and control. The models could be extracted by field-solvers and parasitic extractors. The practical problem is that in VLSI applications(e.g. RF receiver, CPUs, or sensing and actuation chips) those extracted models end up with a very large scale. Thus to efficiently handle those models, approximation of the large scale models is necessary, which is achieved in the presented tool by order reduction. As is always the concern in approximation, a certain degree of accuracy should be satisfied. The other prime concern is stability of the model, because system-level modeling depends highly upon stable building blocks. Another more strict requirement for stability is passivity, arising from the attribute of the physical system.

To fulfill all those requirements, the presented method adopts a projection-based model order reduction method.

The projection matrix is constructed out of the Krylov subspace, with the hope to preserve as much accuracy as it could. For the initially definite and stable model, Galerkin projection could simply preserve the stability. Galerkin projection employs a projection framework by using a left and right projection matrix pair. But in majority of the realistic cases, the initial models are indefinite and unstable, due to either the physical system or the field solver in use for extracting the model. Therefore, Galerkin projection is not able to guarantee the stability of the reduced model.

To guarantee the stability and passivity of the reduced model, the presented work employs a new method which optimizes the degree of accuracy while satisfies the stability and passivity requirements. This is achieved by using a projection framework, namely the left and right projection matrices. At first, this projection framework takes the form of Galerkin projection, aiming to guarantee accuracy. Then, a procedure will optimize over the left projection matrix to fulfill the stability and passivity requirements. These requirements are formed as linear matrix inequalities (LMI), derived from the Lyapunov function of the model. The solution of the LMI will guarantee stability and passivity. Thus the initial model does not have to be stable. In a word, we trade off as little accuracy as possible for stability.

2 Background

A very simple introductory model is given is [Fig.1].

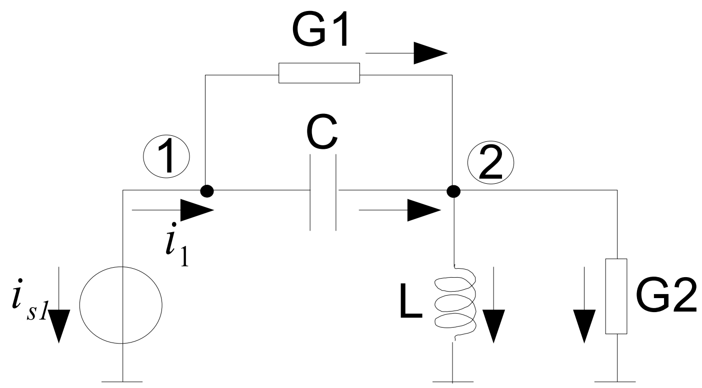

Fig.1 simple RLC system

The electrical behavior of this physical system is modeled by modified nodal analysis (MNA). In MNA, equations based on Kirchhoff's Law could be set up for each non-grounded nodal.

$$
\begin{aligned}
C(\dot{v}_1-\dot{v}_2)+G_1(v_1-v_2)+i_{s1}&=0\\
C(\dot{v}_2-\dot{v}_1)+G_1(v_2-v_1)+G_2 v_2+i_L&=0\\
L\dot{i}_L-v_2&=0\\
-v_1&=-u_{s1}
\end{aligned}
$$

After some transformation and together with the output in concern, the modeled system could be expressed as

$$E\dot{x}=Ax+Bu,\quad y=C^T x \tag{1}$$

The vector x contains the unknowns to be solved.

If the physical system is composed of passive linear elements, such as in this simple RLC circuits, then E and $-(A+A^T)$ are symmetric non-negative definite matrix and the model will be stable and passive.

For the ease of further explanation, define

$$K = A^{-1} E$$
$$R = -A^{-1} B$$

To analyze models in (1), apply Laplace transformation, resulting the transfer function shown in (2).

$$X = (sE - A)^{-1} BU$$
$$Y = C^T X = C^T (sE - A)^{-1} BU$$
$$Tr(s) = C^T (sE - A)^{-1} B \tag{2}$$

At this point, the output response could be fully characterized by this transform function in frequency domain. Though in this example there is only one port, same analysis is applied analogously for multi-port model. The difference would be that B and C will be a matrix instead of a vector.

3 Projection-Based Order Reduction

The order of models described in (1) could be so large that the computational complexity of directly solving them is not affordable. Besides that, such models in practice extracted by field solvers are in majority cases non-stable and non-passive, making them actually not usable. These concerns give rise to find a reduced order model able for the original model. This reduced order model should approximate the original one with a certain degree of accuracy, while guarantee the stability and passivity properties. This order reduction process is implemented in the presented work by a projection-based approach. The basic concepts to be used in the approach will be introduced in the following two parts 3.1 and 3.2.

3.1 Moments Matching and Accuracy

From (2), the accuracy of the reduced model lies in approximating the transfer function, which could be expressed by Taylor expansion around s=0.

$$Tr(s) = M_0 + M_1 s + M_2 s^2 + \ldots$$

The coefficient matrix M_i is defined as the i-th moment and is computed as below.

$$M_i = C^T K^i R$$

Thus, approximating the transfer function could be achieved by matching the lower order moments. Approximation of a matrix could be done by keeping the the most pronounced eigenvalues, giving rise to the difficulty due to eigenvalue decomposition.

3.2 Krylov Subspace and Projection Matrix

In the presented work, the approximation is achieved by projecting the large model to a low order subspace. The projection matrix is constructed with the help of Krylov subspace defined as below.

$$Kr(K, R, q) = colspan[R, KR, K^2 R, \ldots, K^q R]$$

And the basis matrix of this subspace is used as the projection matrix

$$colspan(V) = Kr(K, R, q) \quad , \qquad (3)$$

where q is the order of the reduced model and it is artificially chosen. To be specific, the practical calculation of V is

$$v_k = ((s_p E - A)^{-1} E)^k (s_p E - A)^{-1} B \quad ,$$

corresponding to the k-th vector in V. Worth to be mentioned, this way of constructing the projecting vector is to match the moments around frequency s_p . Similar calculation could be done for matching around another frequency. And since a projection pair (U,V) is to be used, the left projection matrix U could be constructed as in the equation below around different frequency than V.

$$u_k = C^T (s_r E - A)^{-1} ((s_r E - A)^{-1} E)^k$$

4 Two Required Properties

The reduced-order MNA matrix are

$$E_{vv} = V^T E V$$

$$A_{vv} = V^T A V$$

$$B_{v0} = V^T B$$

$$(C^T)_v = C^T V \tag{4}$$

and the reduced model is now expressed by the linear descriptor

$$E_{vv}\dot{z} = A_{vv} z + B_{v0} u, \quad y = (C^T)_v z \ , \tag{5}$$

where $x \approx Vz$.

The reduced model is always passive after the transformation of Galerkin projection in (4) [2]. As for accuracy, the first q moments of the original model could be preserved (the proof is shown in [2], from (27) to (42)), or $(M_{vv})_i = M_i \ , \qquad 0 \le i \le q$.

However, as is mentioned, Galerkin projection requires the original model being stable and definite. In practice, such transformation could not be directly applied in many cases due to the following facts. First, even for stable system, the descriptor might not be SPD, such as in modeling nonlinear systems. Second, even the physical system is stable and passive, the extractor might generate unstable model. The solution is addressed in chapter 5 and 6.

5 Derived Constraints

Each required property implies certain constraints for the reduced order model to satisfy. Interesting enough, these properties are fulfilled through the existence of the Lyapunov function as shown in 5.1. And constraints could be obtained by satisfying the properties of this Lyapunov function as shown in 5.2.

5.1 Lyapunov Function

As is stated in chapter 4, Galerkin projection, able to preserve stability, can not guarantee stability for the extracted unstable large order model. The remedy to this problem is to set up the stability constraints to be fulfilled by the transformation using projection matrix. In the presented work, the algorithm starts with Galerkin projection, meaning the projection framework is achieved by constructing a certain base out of the Krylov subspace. Then, it fixes the right projection matrix and alters the left projection matrix so that the stability constraints are fulfilled.

The constraints are formulated by verifying the existence of Lyapunov function L(x) [3]. Lyapunov function has to satisfy two requirements expressed in (6) and (7).

$$L(x) \geq 0 \ ; \ equal \ only \ when \, x = 0 \tag{6}$$

$$\frac{\partial L(x)}{\partial t} \leq 0 \ ; \ equal \, only \, when \, x = 0 \, (for \, asymptotic \, stability) \tag{7}$$

The construction of Lyapunov function is done by setting it with an artificial form as below.

$$L(x) = x^T E^T P E x$$

5.2 Constraints

Therefore, for satisfying (7),

$$\frac{\partial L(x)}{\partial t} = x^T E^T P E \frac{\partial x}{\partial t} + \frac{\partial x^T}{\partial t} E^T P E x = x^T E^T P E \dot{x} + \dot{x}^T E^T P E x \tag{8}$$

for autonomous system, u=0 and $E\dot{x}$ in (8) could be replaced by Ax , giving rise to the first constraint as below.

$$\frac{\partial L(x)}{\partial t} = x^T E^T P A x + x^T A^T P E x \leq 0$$

Finally the derived constraint is written as

$$E^T P A + A^T P E = -Q \leq 0 \quad ,$$

where Q is a defined symmetric positive semi-definite matrix(SPSD) and $' \leq '$ is the generalized matrix inequality.

Such constraint also has to be true for the reduced system

$$E_{uv} \dot{z} = A_{uv} z + B_v u \, , \quad y = (C^T)_v z \quad ,$$

where $E_{uv}, \ A_{uv}, \ B_v, \ C^T{}_v$ are the transformed system matrices by the projection pair (U,V). Accordingly the constraint is expressed as in (9).

$$E_{uv}^T \hat{P} A_{uv} + A_{uv}^T \hat{P} E_{uv} = -Q \leq 0 \tag{9}$$

The interpretation of constraint (9) is that if there exists an SPD solution $\hat{P}$, then the system is stable. The difficulty in solving for $\hat{P}$ is that this constraint is quadratic in U as shown in (10).

$$V^T E^T U \hat{P} U^T A V + V^T A^T U \hat{P} U^T E V = -Q_2 \tag{10}$$

Since only the existence of solution is of concern, certain simplification could be adopted, provided that it will result in an solution. Such simplification is done by defining $U \hat{P} U^T E V$ as $\hat{U}$. Thus the constraint in (10) becomes

$$\hat{U}^T A V + V^T A^T \hat{U} = -Q_2 \ ,$$

which is linear in $\hat{U}$ and therefore much easier to solve.

After this simplification, the corresponding Lyapunov function of the reduced model takes the form as in (10.1).

$$L(z) = z^T E_{uv}^T P E_{uv} z = z^T V^T E^T U P U^T E V z = z^T \hat{U}^T E V Z \tag{10.1}$$

Due to the other requirement $L(z) > 0$, $\hat{U}^T E V$ has to be an SPD matrix, yielding the other constraint as below.

$$\hat{U}^T E V = Q_1$$

Finally, all the stability constraints have been derived. They are to be satisfied for the reduced model by using a proper projection matrix. Worth to note, even $\hat{U}$ is used for simplification, the constraints expressed by $\hat{U}$ are equal to the original model. Due to this equality, the hat notation will be omitted. For the case of convenience, the stability constraints are written as in (11) from now on.

$$\begin{aligned} U^T A V + V^T A^T U &= -Q_2 \\ U^T E V &= Q_1 \end{aligned} \tag{11}$$

6 Problem Formulation and Solution

The solution of the LMI system is the proper left projection matrix that will guarantee stability. This solution could be obtained by certain LMI solvers [4]. The difficulty lies in the fact that the computational effort for solving the LMI system is proportional to the power of number of unknowns, and the (11) has a very large number of unknowns. Another concern is there exist infinite number of stabilizing subspace spanned by

U. And for one stabilizing subspace, infinite number of U could solve the (11). Thus, the degree of freedom in solving the LMI offers the feasibility to find the most optimal solution in the sense that it is the most accurate projection matrix. Such accuracy is measured by calculating the distance of the solution matrix and the matrix U0 constructed to ensure accuracy. In the presented work from [1], U0 is directly constructed using V, which is actually not the most accurate left projection matrix, because it could well be constructed using the same way as in (3) for matching more moments. As a result of the above two primary concerns, an algorithm is developed based on two corresponding ideas:

I. Fix a majority of unknowns in U and solve for the rest. The benefit is the significantly easier LMI system.

II. Optimize the solution of U for the best accuracy over the LMI constraints.

6.1 The Core Algorithm

I. First certain field solvers could extract the system mode, i.e., the system matrices, giving the original system matrices E, A .

II. Construct the right projection matrix V based on the Krylov subspace approach, as a result the first q moments around certain frequency could be matched. Then assign $U_0 = V$.

III. For the reduced model, its system matrices are formulated as below.

$$E_v = EV \qquad A_v = AV$$
$$\Delta Q_1 = E_v^T U_0 \qquad \Delta Q_2 = V_a^T U_0 + U_0^T V_a$$

Here ΔQ_1 and ΔQ_2 are resulted by substituting the initial U_0 into the constraint formulas and they are not necessarily SPD and therefore yielding an unstable reduced order model.

IV. Next step is to fix the unknowns of U_0 other than p selected rows.

And then select p non-zero rows of the reduced model matrix corresponding to the unknowns to be solved.

$$E_{vp} = sel(E_v, p) \quad A_{vp} = sel(A_v, p)$$

V. The essential optimization process over the stable and passive constraints. The solution is optimal for a certain value of p, but only sub-optimal in the whole solution subspace.

$$\begin{aligned} & min \ \|U_\delta\| \\ & s.t. \ \ U_\delta^T E_{vp} = Q_1 - \Delta Q_1 \\ & \qquad U_\delta^T A_{vp} + U_\delta^T A_{vp}^T = -Q_2 - \Delta Q_2 \\ & \qquad U^T B = V^T C \end{aligned} \tag{12}$$

VI. The final left projection matrix is formulated in (13) by adding the perturbation matrix to the initial matrix constructed for accuracy. The result is a stability preserving but less accurate projection matrix.

$$U = U_0 + \Delta U \tag{13}$$

It needs to be pointed out that in the presented work, the specific optimization algorithm and choice of p is not mentioned.

7 Experimental Results

Consider a very simple 3*3 power grid, consisting of copper wires. Its extracted model (by EMQS solver [5]) is fairly large, with order N=1566. More undesirable is the model is unstable. Applying the core algorithm and setting the reduced order q=10, a reduced stable model is the resulted solution of the optimization process. As is shown by [Fig.2], several interesting observations need to be pointed out.

I. Though being unstable, the Galerkin projection reduced model has an accuracy comparable to the stable reduced model. However, such unstable models would generate unphysical behavior upon certain stimulation such as inputs or noise.

II. A more careful observation of the figure shows that the Galerkin projection generated reduced model actually conforms to the actual curve better than the model generated by the core algorithm, mainly around the peak frequency $f \approx 10^8$. And since the core algorithm starts with Galerkin projection, it adds a small perturbation to the left projection matrix U. As a result, it gives a stable reduced model, with the trade-off of being a little bit less accurate comparing to the Galerkin projection.

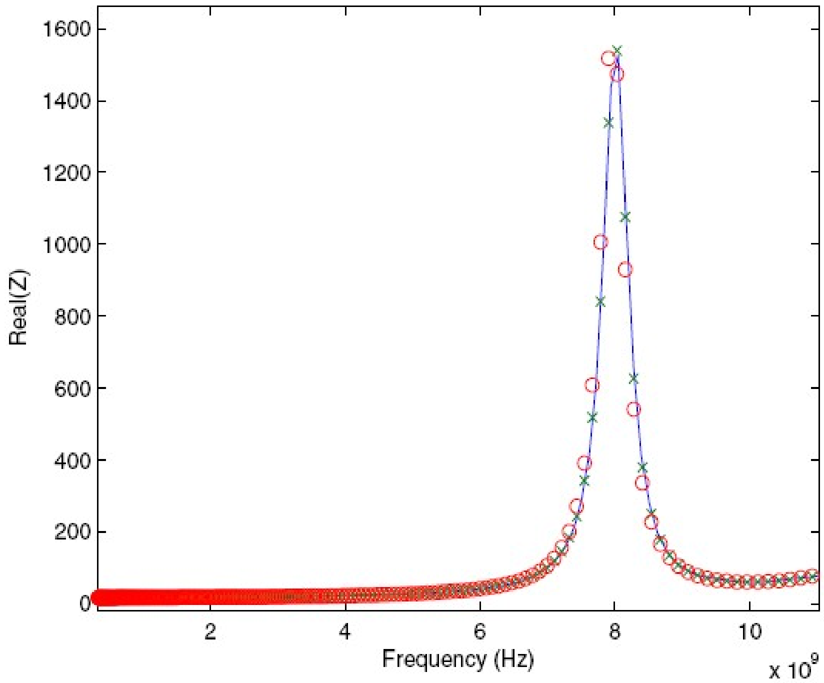

Fig.2 Real part of the impedance for the 3*3 power grid. The original model denoted by solid line is unstable with order N=1566. Galerkin projection returns an unstable reduced model denoted by crosses. The core algorithm gives a stable reduced model denoted by circles

8 Conclusion

The presented work is able to handle indefinite and unstable model in model order reduction. The order reduction is achieved by the projection framework and the stability and passivity properties of the reduced model are satisfied by solving a set of linear constraints. Worth to mention is that same procedure could also be done by fixing the left projection U and optimize for the right projection matrix V. In practice, the whole procedure is implemented by the core algorithm. In this algorithm, in order to make the computational effort independent of the original model size, the number of unknowns to be solved is limited to a very low range simply by fixing all the rest unknowns as the initial values. As a result, a stable but slightly less accurate reduced order model is generated and the specific experimental results show satisfying performance.

References

[1] B.N.Bond,L.Daniel, IEEE Trans,ICCAD,Nov.2008,P.728-735

[2] A.Odabasiglu, M.Celik. PRIMA: passive reduced-order interconnect macromodeling algorithm. IEEE Trans. On CADICS, 1998

[3] M.Vidyagasar. Nonlinear Systems Analysis. Prentice Hall, 1978

[4] J.F.sTURM. Using SeDuMi 1.02, a MATLAB toolbox for optimization over symmetric cones. Optimization Methods and Software, 1999

[5] T.Moselhy, X.Hu, L.Daniel. Pfft in fastmaxwell: a fast impedance extraction solver for 3d conductor structures over substrate. 2007