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Solution Sheet 2
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Fundamental Algorithms

Problem 1 (10 Points)

Consider the following reccurance relation:

$$\begin{aligned} g_1 &= 1 \\ g_2 &= 2 \\ g_n &= g_{n-1} \cdot g_{n-2} \ ; n \geq 3 \end{aligned}$$

What is g_n as a function of fibonacci number? Prove your claim.

Solution

$$\begin{aligned} g_n &= g_{n-1} \cdot g_{n-2} \\ &= (g_{n-2} \cdot g_{n-3}) \cdot g_{n-2} \\ &= g_{n-2}^2 \cdot g_{n-3} \\ &= g_{n-3}^3 \cdot g_{n-4}^2 \\ &\vdots \\ &= g_{n-i}^{F_{i+1}} \cdot g_{n-(i+1)}^{F_i} \\ &= g_{n-(n-2)}^{F_{(n-2)+1}} \cdot g_{n-(n-1)}^{F_{n-2}} \\ &= g_2^{F_{n-1}} \\ &= 2^{F_{n-1}} \end{aligned}$$

This could be proved using induction too.

Problem 2 (10 Points)

Consider the following:

$$\begin{aligned} g_1 &= 1 \\ g_2 &= 1 \\ g_n &= (n-1) \cdot g_{n-1} + (n-2) \cdot g_{n-2} + \ldots + 1 \cdot g_1 \ ; n \geq 3 \end{aligned}$$

What is g_n as a function of n? Prove your claim.

(Extra: Prove: If $g_2 = 2$, then $g_n = \frac{n! \cdot 5}{3!}$)

Solution

$$\begin{aligned}
g_n &= (n-1)\cdot g_{n-1} + (n-2)\cdot g_{n-2} + \ldots + 1\cdot g_1 \\
g_{n-1} &= (n-2)\cdot g_{n-2} + \ldots + 1\cdot g_1 \\
g_n - g_{n-1} &= (n-1)g_{n-1} \\
g_n &= n\cdot g_{n-1}
\end{aligned}$$

Consider expanding the series.

$$\begin{aligned}
g_n &= n\cdot g_{n-1} \\
&= n\cdot(n-1)\cdot g_{n-2} \\
&= n\cdot(n-1)\ldots 3\cdot g(2) \\
&= \frac{n!}{2}
\end{aligned}$$

If $g(2) = 2$ then $g(3) = 5$. That means:

$$\begin{aligned}
g_n &= n\cdot g_{n-1} \\
&= n\cdot(n-1)\cdot g_{n-2} \\
&= n\cdot(n-1)\ldots 4\cdot g(3) \\
&= n\cdot(n-1)\ldots 4\cdot 5 \\
&= \frac{n!}{3!}\cdot 5
\end{aligned}$$

Problem 3 (10 Points)

Give, in Landau notation, the relationships between every pair of the following functions. n, $\lg n$, n^2, $n\lg n$ and 2^n.

Solution

Let's tabulate them together.

	n	$\lg n$	n^2	$n\lg n$	2^n
n	Θ	ω	o	o	o
$\lg n$	o	Θ	o	o	o
n^2	ω	ω	Θ	ω	o
$n\lg n$	ω	ω	o	Θ	o
2^n	ω	ω	ω	ω	Θ

The following picture shows the growth rate of each functions Blue $= n$, Green $= n^2$, Black $= \ln n$, Red $= n \ln n$ and Yellow $= 2^n$. In the figure, n^2 seems to grow faster than 2^n. In reality 2^n over grows n^2 for $n > 4$.