21 Weighted Bipartite Matching

Weighted Bipartite Matching/Assignment

- ▶ Input: undirected, bipartite graph $G = L \cup R, E$.
- ▶ an edge $e = (\ell, r)$ has weight $w_e \geq 0$
- ▶ find a matching of maximum weight, where the weight of a matching is the sum of the weights of its edges

Simplifying Assumptions (wlog [why?]):

- ▶ assume that $|L| = |R| = n$
- ▶ assume that there is an edge between every pair of nodes $(\ell, r) \in V \times V$

Weighted Bipartite Matching

Theorem 97 (Halls Theorem)

A bipartite graph $G = (L \cup R, E)$ has a perfect matching if and only if for all sets $S \subseteq L$, $|\Gamma(S)| \geq |S|$, where $\Gamma(S)$ denotes the set of nodes in R that have a neighbour in S.

Halls Theorem

Proof:

⇐ Of course, the condition is necessary as otherwise not all nodes in S could be matched to different neigbhours.

⇒ For the other direction we need to argue that the minimum cut in the graph G' is at least $|L|$.

- Let S denote a minimum cut and let $L_S \overset{\text{def}}{=} L \cap S$ and $R_S \overset{\text{def}}{=} R \cap S$ denote the portion of S inside L and R, respectively.
- Clearly, all neighbours of nodes in L_S have to be in S, as otherwise we would cut an edge of infinite capacity.
- This gives $R_S \geq |\Gamma(L_S)|$.
- The size of the cut is $|L| - |L_S| + |R_S|$.
- Using the fact that $|\Gamma(L_S)| \geq L_S$ gives that this is at least $|L|$.

Algorithm Outline

Idea:

We introduce a node weighting $\vec{x}$. Let for a node $v \in V$, $x_v \geq 0$ denote the weight of node v.

- ▶ Suppose that the node weights dominate the edge-weights in the following sense:

$$x_u + x_v \geq w_e \text{ for every edge } e = (u, v).$$

- ▶ Let $H(\vec{x})$ denote the subgraph of G that only contains edges that are tight w.r.t. the node weighting $\vec{x}$, i.e. edges $e = (u, v)$ for which $w_e = (u, v)$.

- ▶ Try to compute a perfect matching in the subgraph $H(\vec{x})$. If you are successful you found an optimal matching.

Algorithm Outline

Reason:

▶ The weight of your matching M^* is

$$\sum_{(u,v) \in M^*} w_{(u,v)} = \sum_{(u,v) \in M^*} (x_u + x_v) = \sum_v x_v \ .$$

▶ Any other matching M has

$$\sum_{(u,v) \in M} w_{(u,v)} \leq \sum_{(u,v) \in M} (x_u + x_v) \leq \sum_v x_v \ .$$

Algorithm Outline

What if you don't find a perfect matching?

Then, Halls theorem guarantees you that there is a set $S \subseteq L$, with $|\Gamma(S)| < |S|$, where Γ denotes the neighbourhood w.r.t. the subgraph $H(\vec{x})$.

Idea: reweight such that:

- ▶ the total weight assigned to nodes decreases
- ▶ the weight function still dominates the edge-weights

If we can do this we have an algorithm that terminates with an optimal solution (we analyze the running time later).

Changing Node Weights

Increase node-weights in $\Gamma(S)$ by $+\delta$, and decrease the node-weights in S by $-\delta$.
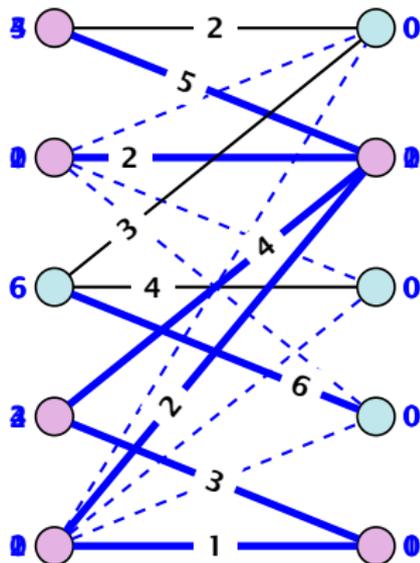
- ▶ Total node-weight decreases.
- ▶ Only edges from S to $R - \Gamma(S)$ decrease in their weight.
- ▶ Since, none of these edges is tight (otw. the edge would be contained in $H(\vec{x})$, and hence would go between S and $\Gamma(S)$) we can do this decrement for small enough $\delta > 0$ until a new edge gets tight.
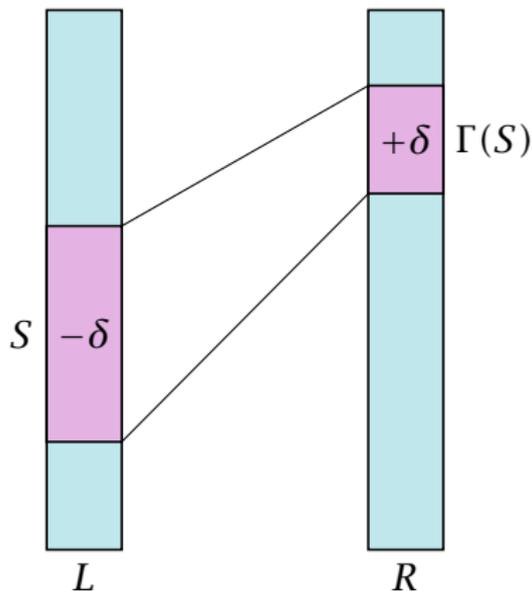
Weighted Bipartite Matching

Edges not drawn have weight 0.

$\delta = 1$ $\delta = 1$

Analysis

How many iterations do we need?

▶ One reweighting step increases the number of edges out of S by at least one.

▶ Assume that we have a maximum matching that saturates the set $\Gamma(S)$, in the sense that every node in $\Gamma(S)$ is matched to a node in S (we will show that we can always find S and a matching such that this holds).

▶ This matching is still contained in the new graph, because all its edges either go between $\Gamma(S)$ and S or between $L - S$ and $R - \Gamma(S)$.

▶ Hence, reweighting does not decrease the size of a maximum matching in the tight sub-graph.

Analysis

- ▶ We will show that after at most n reweighting steps the size of the maximum matching can be increased by finding an augmenting path.

- ▶ This gives a polynomial running time.

Analysis

How do we find S?

- ▶ Start on the left and compute an alternating tree, starting at any free node u.
- ▶ If this construction stops, there is no perfect matching in the tight subgraph (because for a perfect matching we need to find an augmenting path starting at u).
- ▶ The set of even vertices is on the left and the set of odd vertices is on the right and contains all neighbours of even nodes.
- ▶ All odd vertices are matched to even vertices. Furthermore, the even vertices additionally contain the free vertex u. Hence, $|V_{\text{odd}}| = |\Gamma(V_{\text{even}})| < |V_{\text{even}}|$, and all odd vertices are saturated in the current matching.

Analysis

- The current matching does not have any edges from V_{odd} to outside of $L \setminus V_{even}$ (edges that may possibly deleted by changing weights).

- After changing weights, there is at least one more edge connecting V_{even} to a node outside of V_{odd}. After at most n reweights we can do an augmentation.

- A reweighting can be trivially performed in time $\mathcal{O}(n^2)$ (keeping track of the tight edges).

- An augmentation takes at most $\mathcal{O}(n)$ time.

- In total we otain a running time of $\mathcal{O}(n^4)$.

- A more careful implementation of the algorithm obtains a running time of $\mathcal{O}(n^3)$.