7.3 AVL-Trees

Definition 15

AVL-trees are binary search trees that fulfill the following balance condition. For every node v

$$|\mathrm{height}(\text{left sub-tree}(v)) - \mathrm{height}(\text{right sub-tree}(v))| \leq 1 \ .$$

Lemma 16

An AVL-tree of height h contains at least $F_{h+2} - 1$ and at most $2^h - 1$ internal nodes, where F_n is the n-th Fibonacci number ($F_0 = 0$, $F_1 = 1$), and the height is the maximal number of edges from the root to an (empty) dummy leaf.

Proof.

The upper bound is clear, as a binary tree of height h can only contain

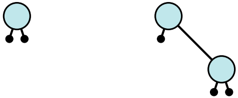
$$\sum_{j=0}^{h-1} 2^j = 2^h - 1$$

internal nodes.

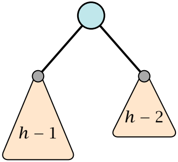
Proof (cont.)

Induction (base cases):

1. an AVL-tree of height $h = 1$ contains at least one internal node, $1 \geq F_3 - 1 = 2 - 1 = 1$.
2. an AVL tree of height $h = 2$ contains at least two internal nodes, $2 \geq F_4 - 1 = 3 - 1 = 2$



Induction step:
An AVL-tree of height $h \geq 2$ of minimal size has a root with sub-trees of height $h-1$ and $h-2$, respectively. Both, sub-trees have minmal node number.



Let

$$f_h := 1 + \text{minimal size of AVL-tree of height } h \;.$$

Then

$$\begin{aligned} f_1 &= 2 && = F_3 \\ f_2 &= 3 && = F_4 \\ f_h - 1 &= 1 + f_{h-1} - 1 + f_{h-2} - 1\,, && \text{hence} \\ f_h &= f_{h-1} + f_{h-2} && = F_{h+2} \end{aligned}$$

7.3 AVL-Trees

Since

$$F(k) \approx \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^k \quad ,$$

an AVL-tree with n internal nodes has height $\Theta(\log n)$.

7.3 AVL-Trees

We need to maintain the balance condition through rotations.

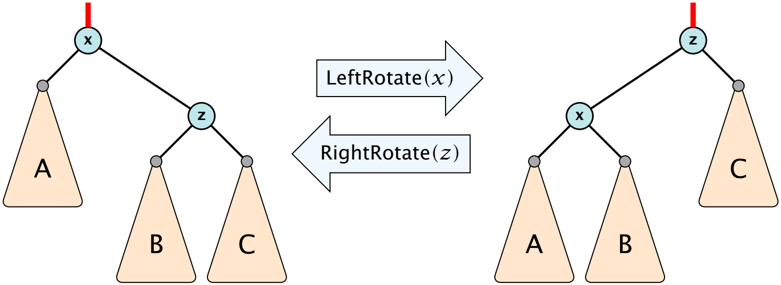
For this we store in every internal tree-node v the balance of the node. Let v denote a tree node with left child c_ℓ and right child c_r.

$$\mathrm{balance}[v] := \mathrm{height}(T_{c_\ell}) - \mathrm{height}(T_{c_r}) \ ,$$

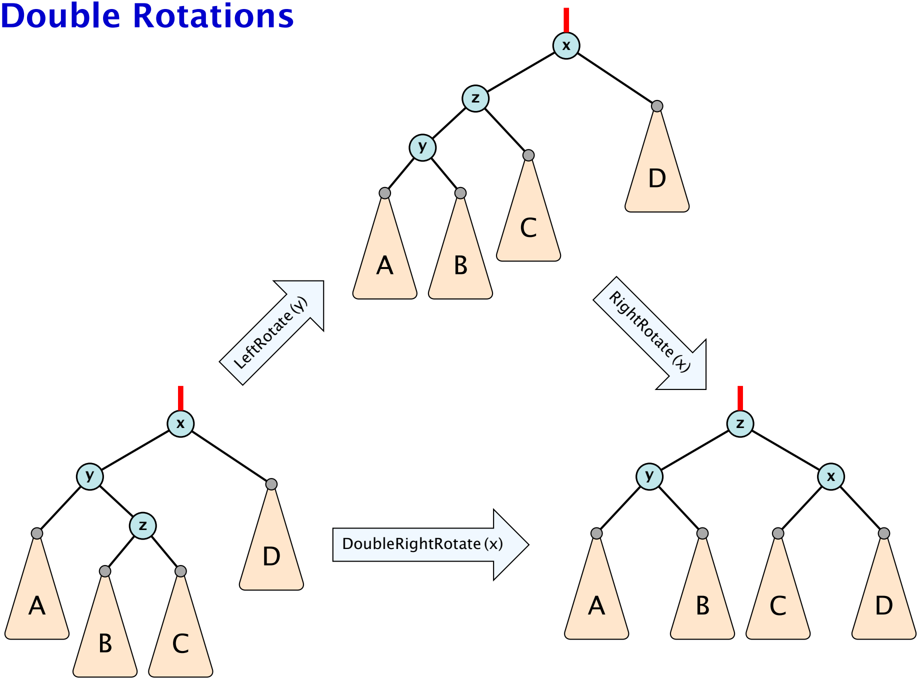
where T_{c_ℓ} and T_{c_r}, are the sub-trees rooted at c_ℓ and c_r, respectively.

Rotations

The properties will be maintained through rotations:

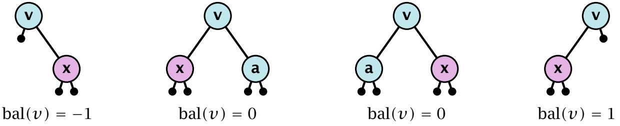


Double Rotations



AVL-trees: Insert

- Insert like in a binary search tree.
- Let v denote the parent of the newly inserted node x.
- One of the following cases holds:



- If $\text{bal}[v] \neq 0$, T_v has changed height; the balance-constraint may be violated at ancestors of v.
- Call fix-up$(\text{parent}[v])$ to restore the balance-condition.

AVL-trees: Insert

Invariant at the beginning fix-up(v):

1. The balance constraints holds at all descendants of v.
2. A node has been inserted into T_c, where c is either the right or left child of v.
3. T_c has increased its height by one (otw. we would already have aborted the fix-up procedure).
4. The balance at the node c fulfills $\text{balance}[c] \in \{-1, 1\}$. This holds because if the balance of c is 0, then T_c did not change its height, and the whole procedure will have been aborted in the previous step.

AVL-trees: Insert

Algorithm 11 AVL-fix-up-insert(v)

if balance$[v] \in \{-2, 2\}$ **then** DoRotationInsert(v);
if balance$[v] \in \{0\}$ **return**;
AVL-fix-up-insert(parent$[v]$);

We will show that the above procedure is correct, and that it will do at most one rotation.

AVL-trees: Insert

Algorithm 12 DoRotationInsert(v)

```
if balance[v] = −2 then
    if balance[right[v]] = −1 then
        LeftRotate(v);
    else
        DoubleLeftRotate(v);
else
    if balance[left[v]] = 1 then
        RightRotate(v);
    else
        DoubleRightRotate(v);
```

AVL-trees: Insert

It is clear that the invariant for the fix-up routine holds as long as no rotations have been done.

We have to show that after doing one rotation **all** balance constraints are fulfilled.

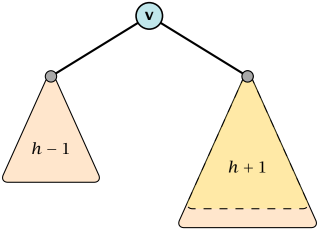
We show that after doing a rotation at v:

- v fulfills balance condition.
- All children of v still fulfill the balance condition.
- The height of T_v is the same as before the insert-operation took place.

We only look at the case where the insert happened into the right sub-tree of v. The other case is symmetric.

AVL-trees: Insert

We have the following situation:

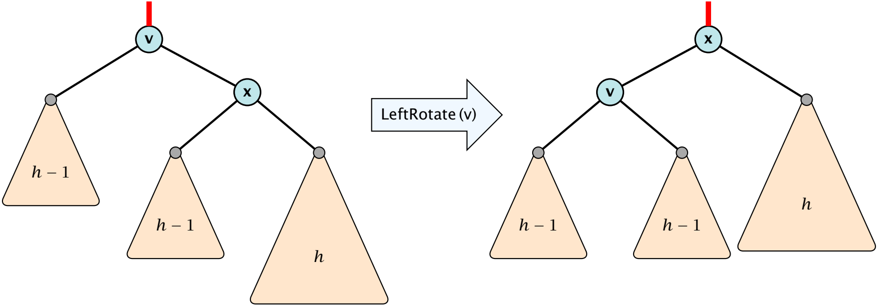


The right sub-tree of v has increased its height which results in a balance of -2 at v.

Before the insertion the height of T_v was $h+1$.

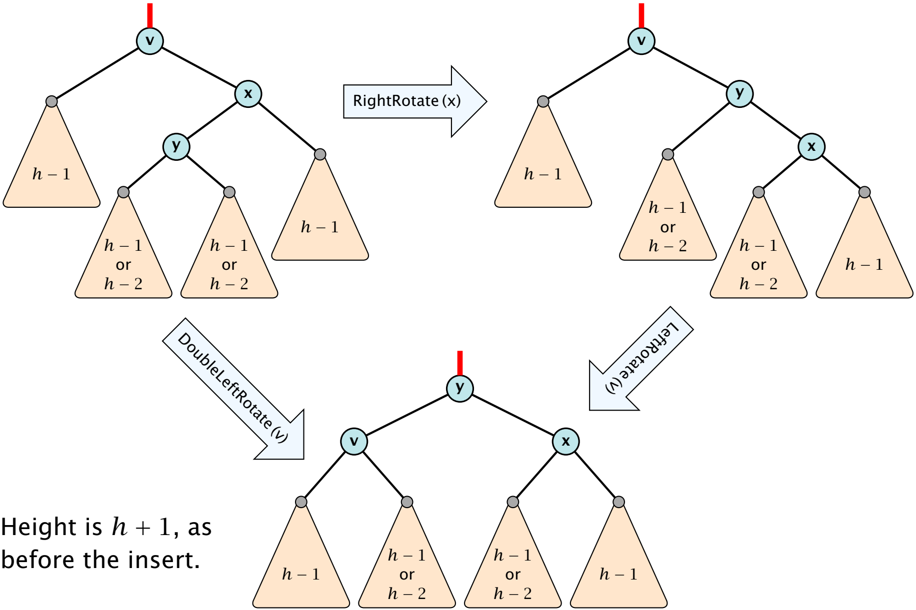
Case 1: balance[right[v]] = −1

We do a left rotation at v



Now, T_v has height $h + 1$ as before the insertion. Hence, we do not need to continue.

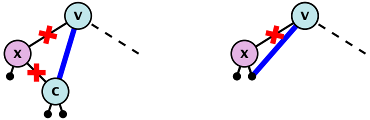
Case 2: balance[right[v]] = 1



Height is $h + 1$, as before the insert.

AVL-trees: Delete

- Delete like in a binary search tree.
- Let v denote the parent of the node that has been spliced out.
- The balance-constraint may be violated at v, or at ancestors of v, as a sub-tree of a child of v has reduced its height.
- Initially, the node c—the new root in the sub-tree that has changed— is either a dummy leaf or a node with two dummy leafs as children.



Case 1 Case 2

In both cases $\text{bal}[c] = 0$.

- Call fix-up(v) to restore the balance-condition.

AVL-trees: Delete

Invariant at the beginning fix-up(v):

1. The balance constraints holds at all descendants of v.
2. A node has been deleted from T_c, where c is either the right or left child of v.
3. T_c has either decreased its height by one or it has stayed the same (note that this is clear right after the deletion but we have to make sure that it also holds after the rotations done within T_c in previous iterations).
4. The balance at the node c fulfills $\text{balance}[c] = \{0\}$. This holds because if the balance of c is in $\{-1, 1\}$, then T_c did not change its height, and the whole procedure will have been aborted in the previous step.

AVL-trees: Delete

Algorithm 13 AVL-fix-up-delete(v)

if balance[v] $\in \{-2, 2\}$ **then** DoRotationDelete(v);
if balance[v] $\in \{-1, 1\}$ **return**;
AVL-fix-up-delete(parent[v]);

We will show that the above procedure is correct. However, for the case of a delete there may be a logarithmic number of rotations.

AVL-trees: Delete

Algorithm 14 DoRotationDelete(v)

```
if balance[v] = −2 then
    if balance[right[v]] = −1 then
        LeftRotate(v);
    else
        DoubleLeftRotate(v);
else
    if balance[left[v]] = {0, 1} then
        RightRotate(v);
    else
        DoubleRightRotate(v);
```

AVL-trees: Delete

It is clear that the invariant for the fix-up routine holds as long as no rotations have been done.

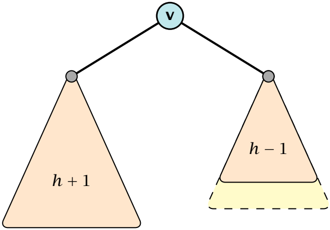
We show that after doing a rotation at v:

- v fulfills balance condition.
- All children of v still fulfill the balance condition.
- If now $\text{balance}[v] \in \{-1, 1\}$ we can stop as the height of T_v is the same as before the deletion.

We only look at the case where the deleted node was in the right sub-tree of v. The other case is symmetric.

AVL-trees: Delete

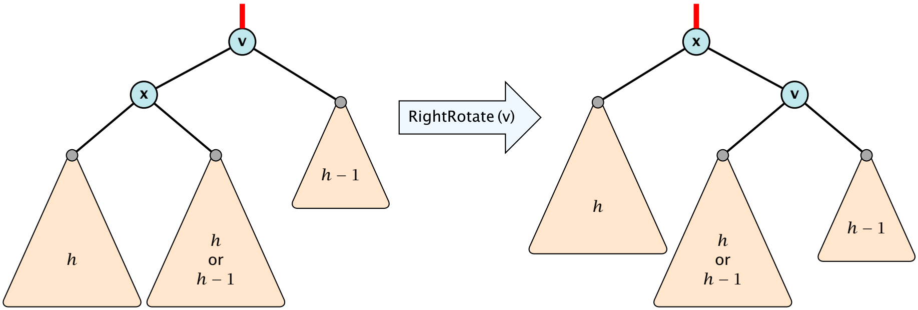
We have the following situation:



The right sub-tree of v has decreased its height which results in a balance of 2 at v.

Before the insertion the height of T_v was $h + 2$.

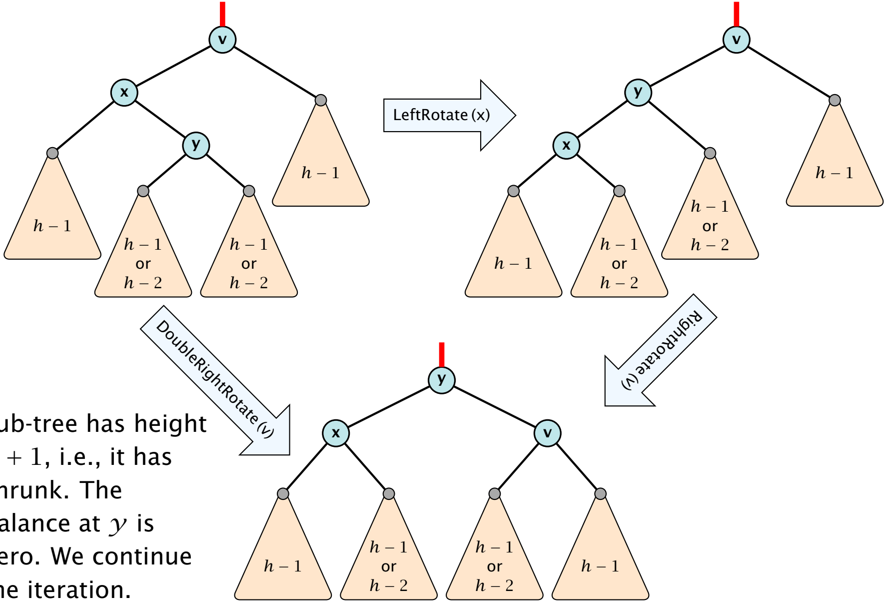
Case 1: balance[left[v]] $\in$ {0, 1}



If the middle subtree has height h the whole tree has height $h + 2$ as before the deletion. The iteration stops as the balance at the root is non-zero.

If the middle subtree has height $h - 1$ the whole tree has decreased its height from $h + 2$ to $h + 1$. We do continue the fix-up procedure as the balance at the root is zero.

Case 2: balance[left[v]] = −1



Sub-tree has height $h + 1$, i.e., it has shrunk. The balance at y is zero. We continue the iteration.