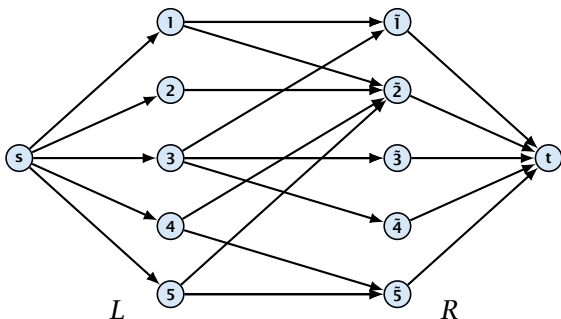


## 17 Bipartite Matching via Flows

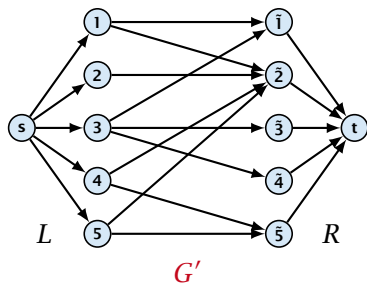
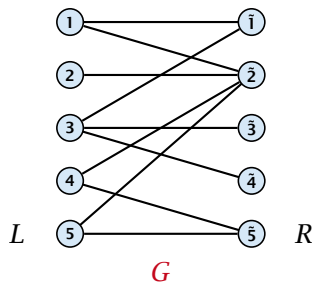
- ▶ Input: undirected, bipartite graph  $G = (L \uplus R \uplus \{s, t\}, E')$ .
- ▶ Direct all edges from  $L$  to  $R$ .
- ▶ Add source  $s$  and connect it to all nodes on the left.
- ▶ Add  $t$  and connect all nodes on the right to  $t$ .
- ▶ All edges have unit capacity.



# Proof

## Max cardinality matching in $G \leq$ value of maxflow in $G'$

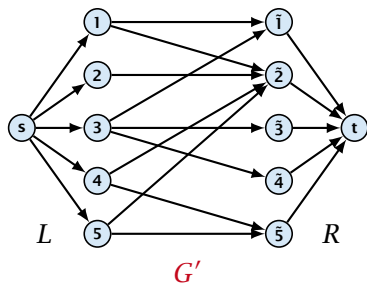
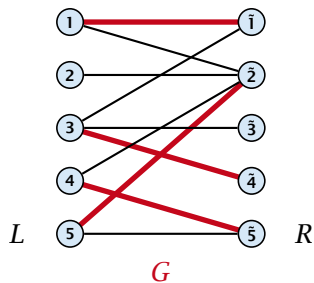
- ▶ Given a maximum matching  $M$  of cardinality  $k$ .
- ▶ Consider flow  $f$  that sends one unit along each of  $k$  paths.
- ▶  $f$  is a flow and has cardinality  $k$ .



# Proof

## Max cardinality matching in $G \leq$ value of maxflow in $G'$

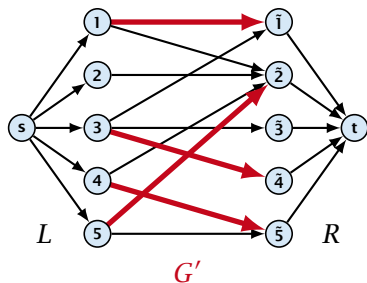
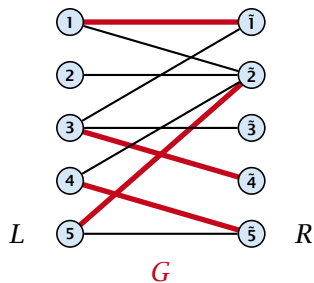
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# Proof

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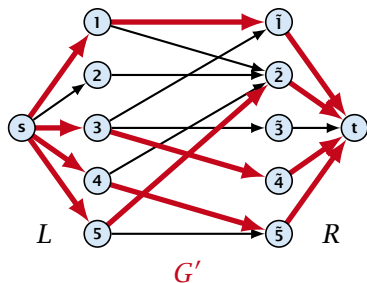
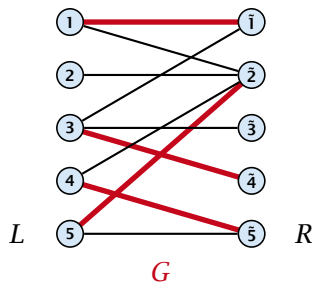
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# Proof

## Max cardinality matching in $G \leq$ value of maxflow in $G'$

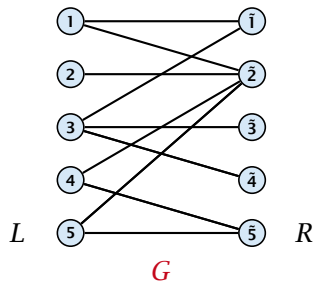
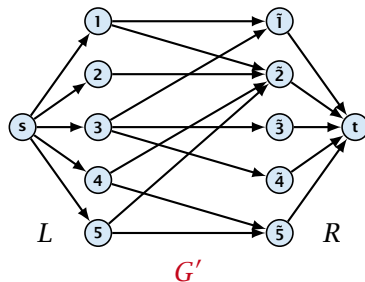
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# Proof

## Max cardinality matching in $G \geq$ value of maxflow in $G'$

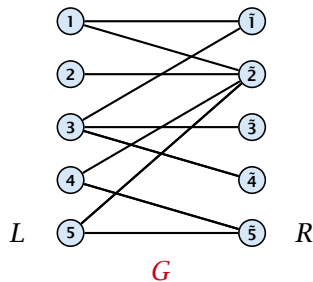
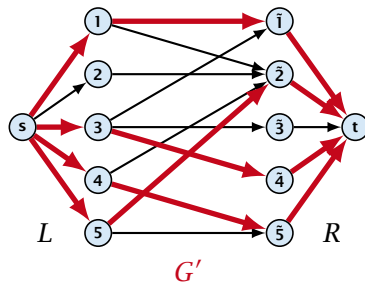
- ▶ Let  $f$  be a maxflow in  $G'$  of value  $k$
- ▶ Integrality theorem  $\Rightarrow k$  integral; we can assume  $f$  is 0/1.
- ▶ Consider  $M =$  set of edges from  $L$  to  $R$  with  $f(e) = 1$ .
- ▶ Each node in  $L$  and  $R$  participates in at most one edge in  $M$ .
- ▶  $|M| = k$ , as the flow must use at least  $k$  middle edges.



# Proof

## Max cardinality matching in $G \geq$ value of maxflow in $G'$

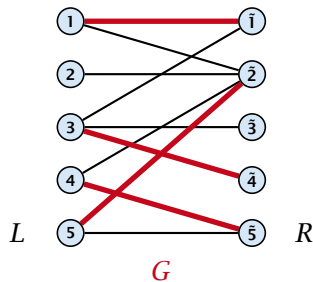
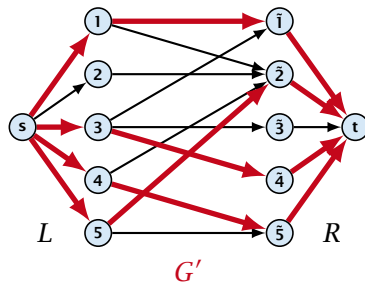
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# Proof

## Max cardinality matching in $G \geq$ value of maxflow in $G'$

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- ▶  $|M| = k$ , as the flow must use at least  $k$  middle edges.





# 17 Bipartite Matching via Flows

## Which flow algorithm to use?

- ▶ Generic augmenting path:  $\mathcal{O}(m \text{val}(f^*)) = \mathcal{O}(mn)$ .
- ▶ Capacity scaling:  $\mathcal{O}(m^2 \log C) = \mathcal{O}(m^2)$ .