Preflows

Definition 1
An (s,t)-preflow is a function $f : E \mapsto \mathbb{R}^+$ that satisfies

1. For each edge e
$$0 \le f(e) \le c(e) \; .$$
(capacity constraints)

2. For each $v \in V \setminus \{s,t\}$
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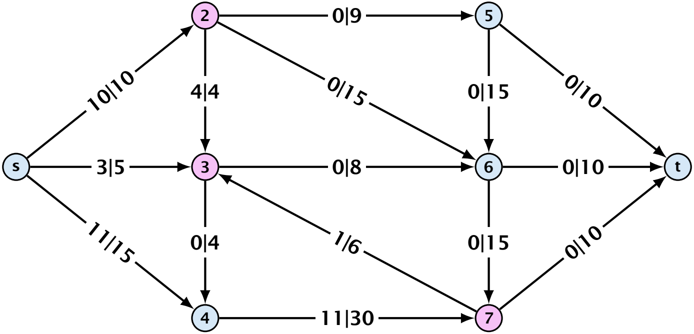
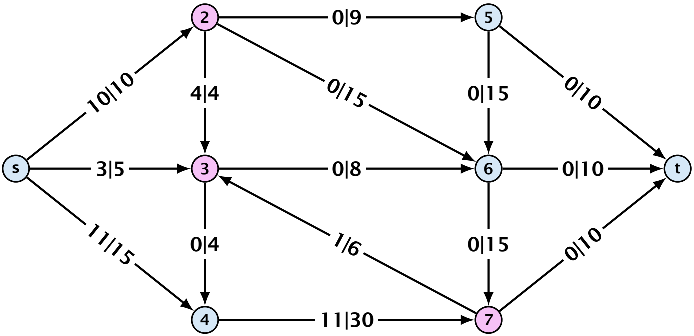
Preflows

Example 2



Preflows

Example 2



A node that has $\sum_{e \in \text{out}(v)} f(e) < \sum_{e \in \text{into}(v)} f(e)$ is called an active node.

Preflows

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Definition:
A labelling is a function $\ell : V \to \mathbb{N}$. It is valid for preflow f if

- $\ell(u) \le \ell(v) + 1$ for all edges in the residual graph G_f (only non-zero capacity edges!!!)

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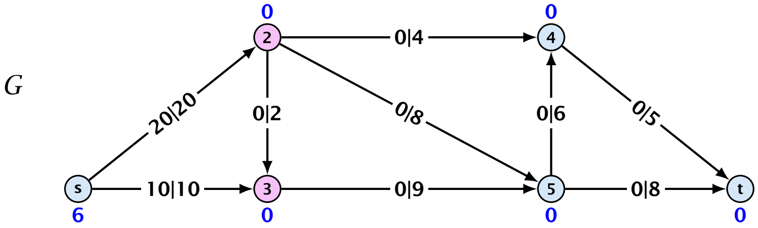
Intuition:
The labelling can be viewed as a height function. Whenever the height from node u to node v decreases by more than 1 (i.e., it goes very steep downhill from u to v), the corresponding edge must be saturated.

Preflows



G
0
0
2
0|4
4
20|20
0|2
0|8
0|6
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s
10|10
3
0|9
5
0|8
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6
0
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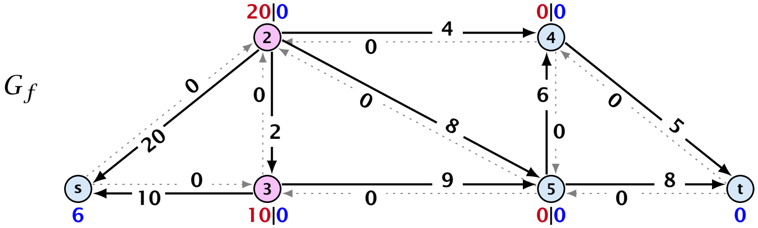
0



G_f
20|0
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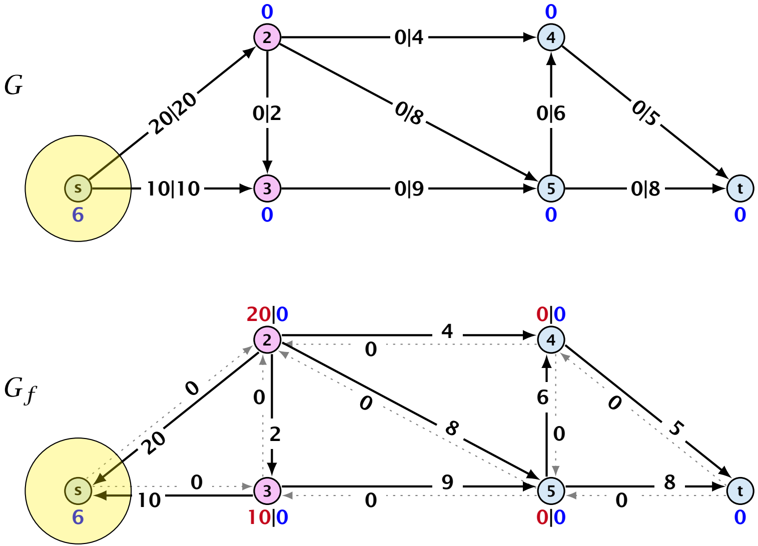
10|0
0|0
0

Preflows


G
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Preflows

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Lemma 3

A preflow that has a valid labelling saturates a cut.

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Lemma 4

A flow that has a valid labelling is a maximum flow.

Push Relabel Algorithms

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- start with some preflow and some valid labelling

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- start with some preflow and some valid labelling
- successively change the preflow while maintaining a valid labelling
- stop when you have a flow (i.e., no more active nodes)

Changing a Preflow

An arc (u, v) with $c_f(u, v) > 0$ in the residual graph is admissable if $\ell(u) = \ell(v) + 1$ (i.e., it goes downwards w.r.t. labelling ℓ).

The push operation
Consider an active node u with excess flow $f(u) = \sum_{e \in \text{into}(u)} f(e) - \sum_{e \in \text{out}(u)} f(e)$ and suppose $e = (u, v)$ is an admissable arc with residual capacity $c_f(e)$.

We can send flow $\min\{c_f(e), f(u)\}$ along e and obtain a new preflow. The old labelling is still valid (!!!).

- saturating push: $\min\{f(u), c_f(e)\} = c_f(e)$
  the arc e is deleted from the residual graph
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Push Relabel Algorithms

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The relabel operation

Consider an active node u that does not have an outgoing admissable arc.

Increasing the label of u by 1 results in a valid labelling.

- Edges (w, u) incoming to u still fulfill their constraint $\ell(w) \leq \ell(u) + 1$.
- An outgoing edge (u, w) had $\ell(u) < \ell(w) + 1$ before since it was not admissable. Now: $\ell(u) \leq \ell(w) + 1$.

Push Relabel Algorithms

Intuition:
We want to send flow downwards, since the source has a height/label of n and the target a height/label of 0. If we see an active node u with an admissible arc we push the flow at u towards the other end-point that has a lower height/label. If we do not have an admissible arc but excess flow into u it should roughly mean that the level/height/label of u should rise. (If we consider the flow to be water than this would be natural).

Note that the above intuition is very incorrect as the labels are integral, i.e., they cannot really be seen as the height of a node.

Reminder

- In a preflow nodes may not fulfill conserveration constraints but a node may have more incoming flow than outgoing flow.
- Such a node is called active.
- A labelling is valid if for every edge (u,v) in the residual graph $\ell(u) \le \ell(v) + 1$.
- An arc (u,v) in residual graph is admissable if $\ell(u) = \ell(v) + 1$.
- A saturation push along e pushes an amount of $c(e)$ flow along the edge, thereby saturating the edge (and making it dissappear from the residual graph).
- A non-saturating push along $e = (u,v)$ pushes a flow of $f(u)$, where $f(u)$ is the excess flow of u. This makes u inactive.

Push Relabel Algorithms

Algorithm 46 maxflow(G, s, t, c)

1: find initial preflow f
2: **while** there is active node u **do**
3:     **if** there is admiss. arc e out of u **then**
4:         push(G, e, f, c)
5:     **else**
6:         relabel(u)
7: **return** f

Push Relabel Algorithms

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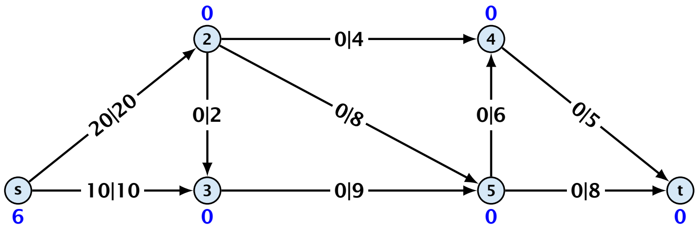
```

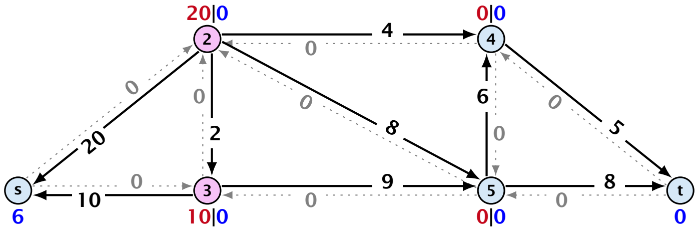
In the following example we always stick to the same active node u until it becomes inactive but this is not required.

Preflow Push Algorithm

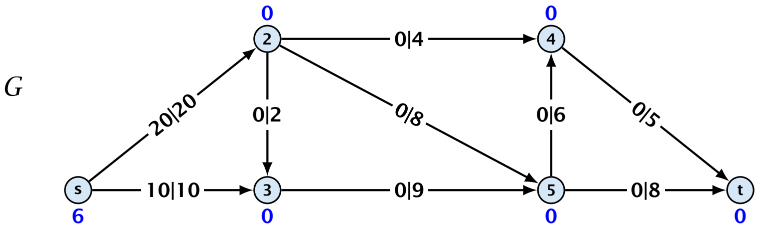
G

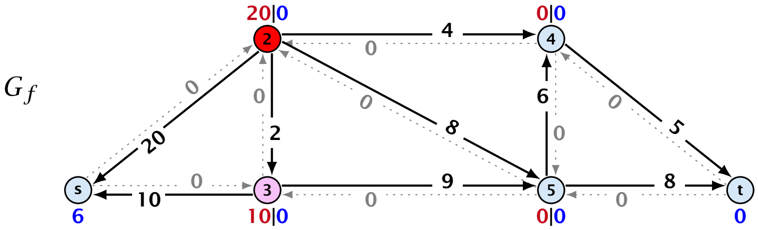


G_f



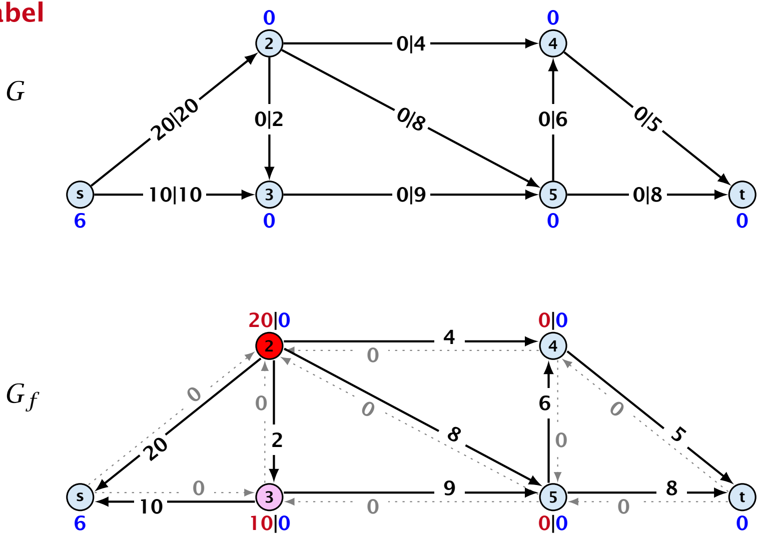
Preflow Push Algorithm



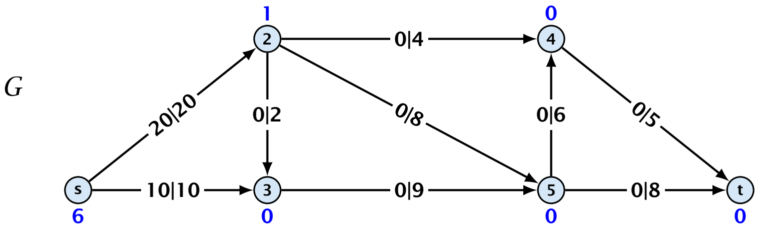


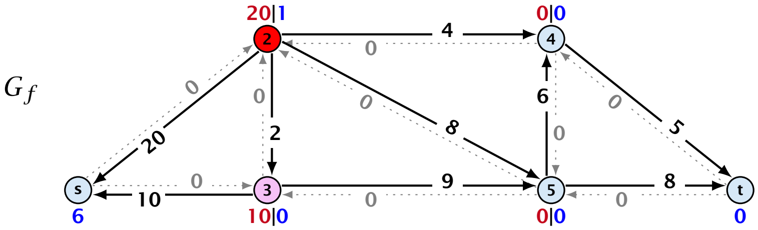
Preflow Push Algorithm

relabel



Preflow Push Algorithm

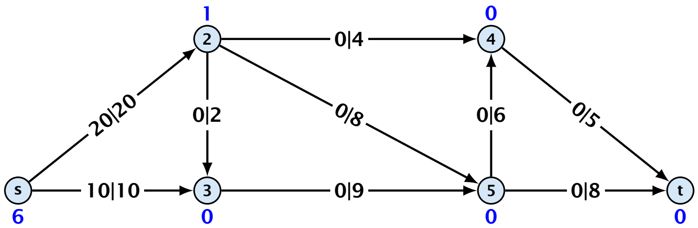




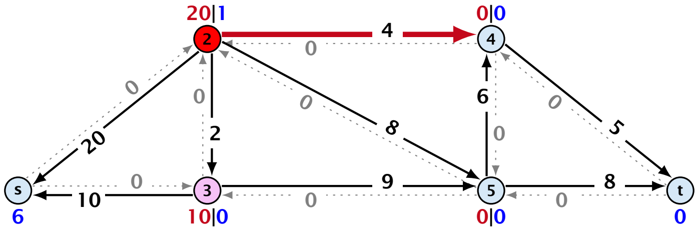
Preflow Push Algorithm

push

G



G_f



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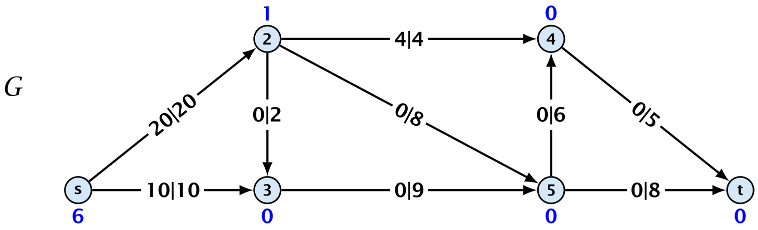
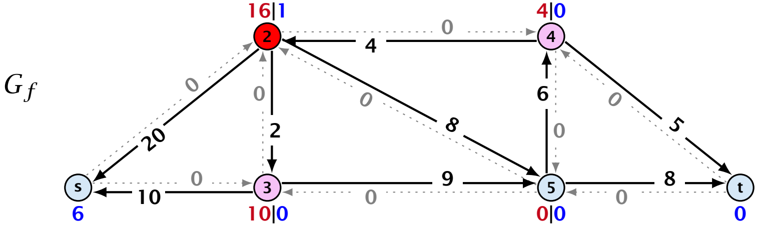
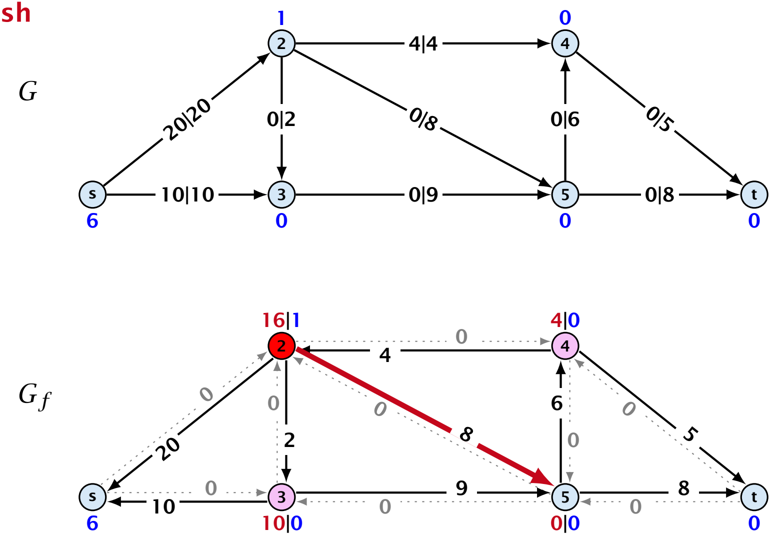


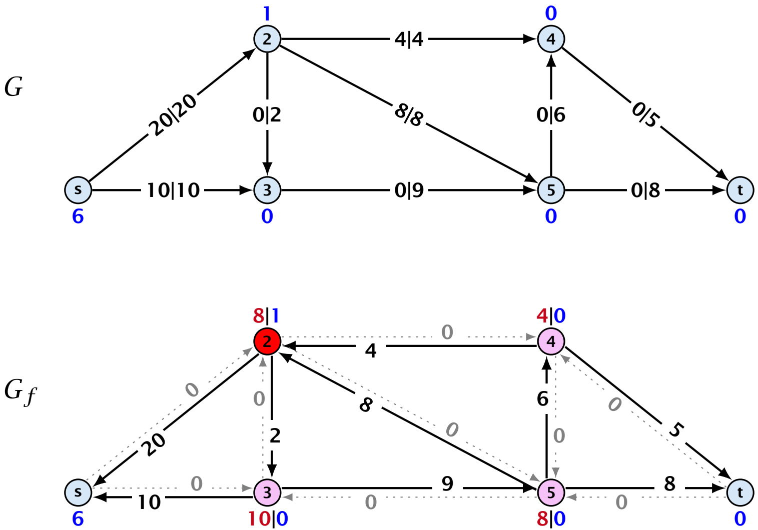


Preflow Push Algorithm

push

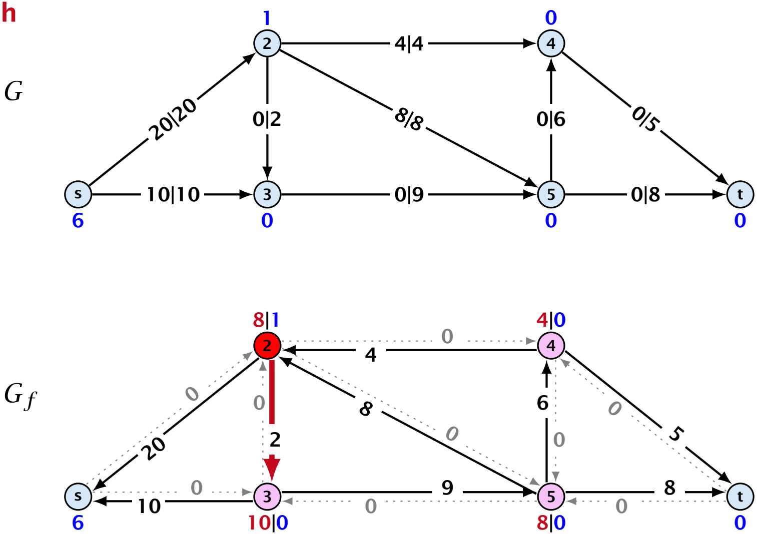


Preflow Push Algorithm

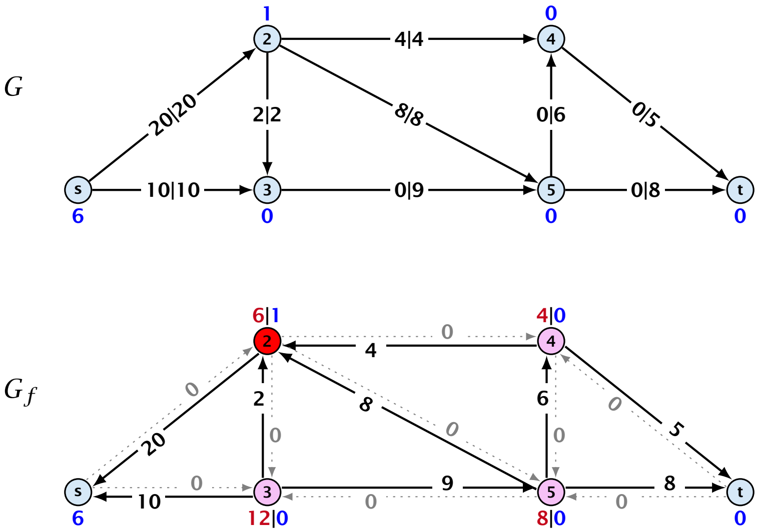


Preflow Push Algorithm

push

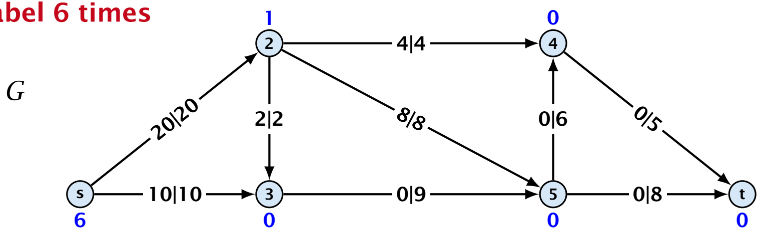


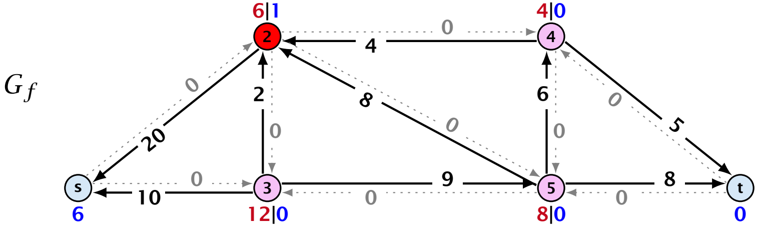
Preflow Push Algorithm



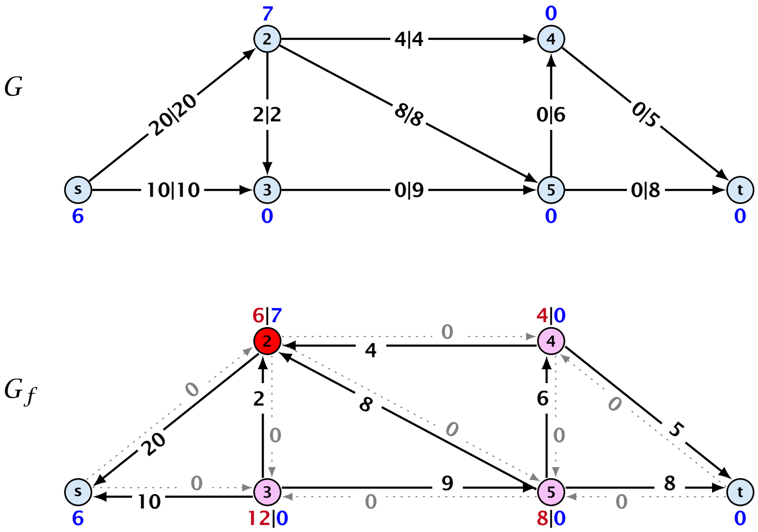
Preflow Push Algorithm

relabel 6 times





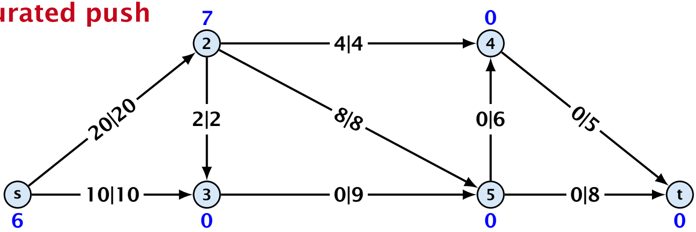
Preflow Push Algorithm



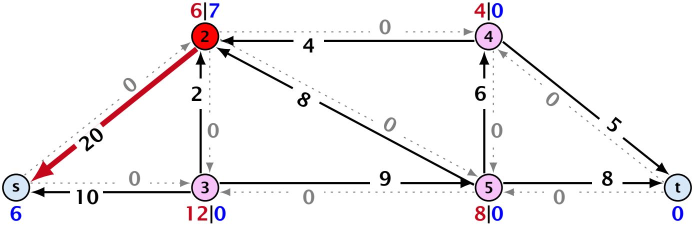
Preflow Push Algorithm

non-saturated push

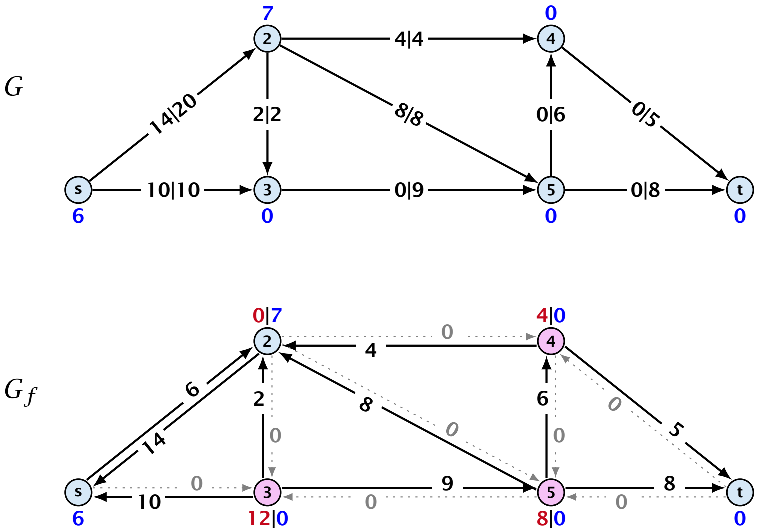
G



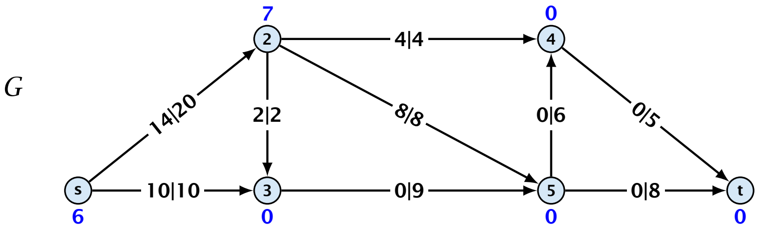
G_f

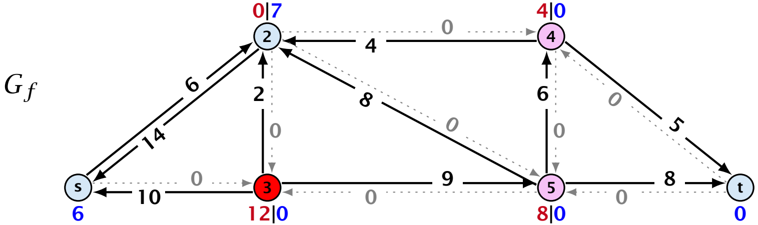


Preflow Push Algorithm



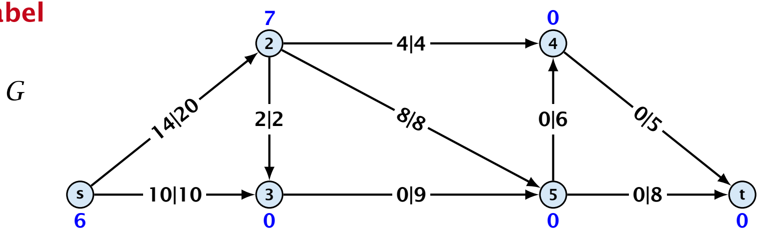
Preflow Push Algorithm

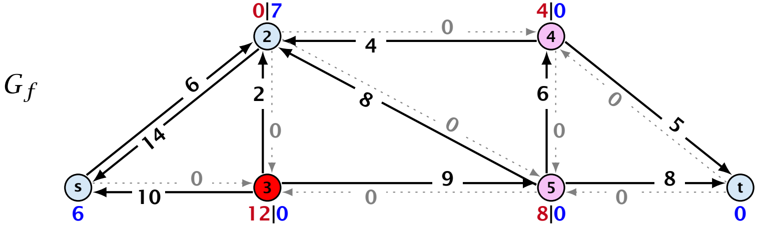




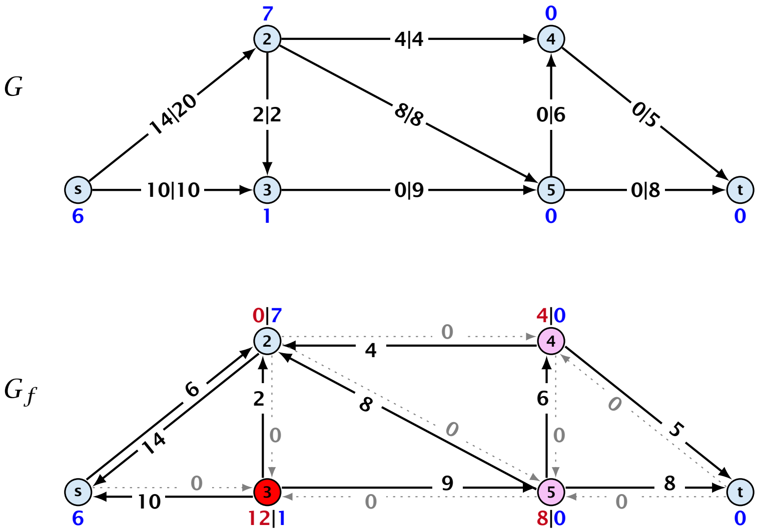
Preflow Push Algorithm

relabel





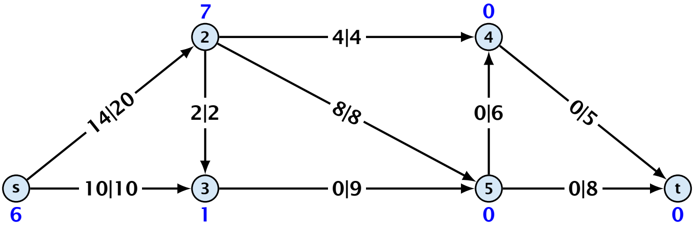
Preflow Push Algorithm



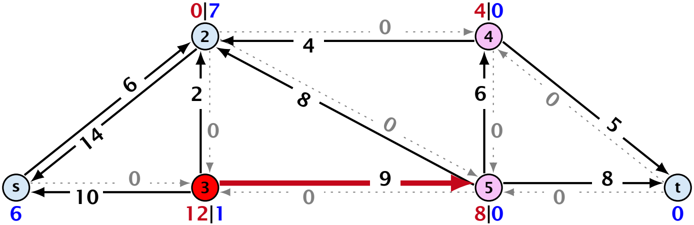
Preflow Push Algorithm

push

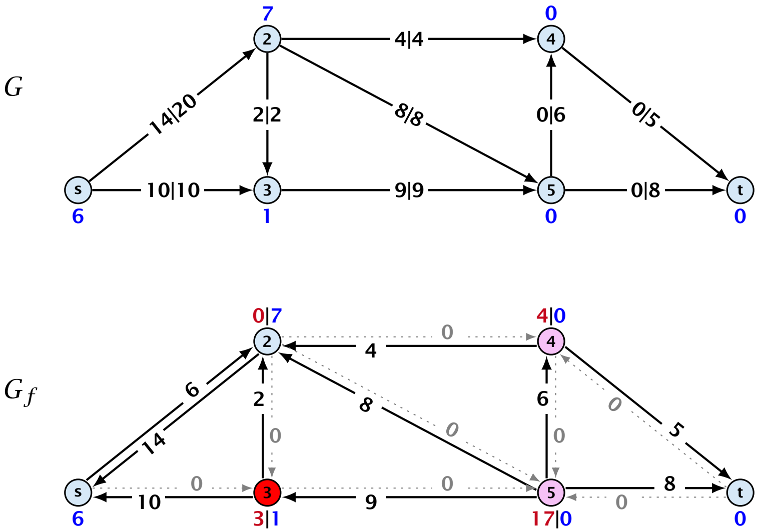
G



G_f

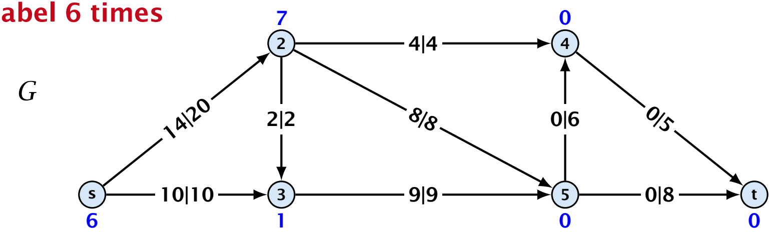


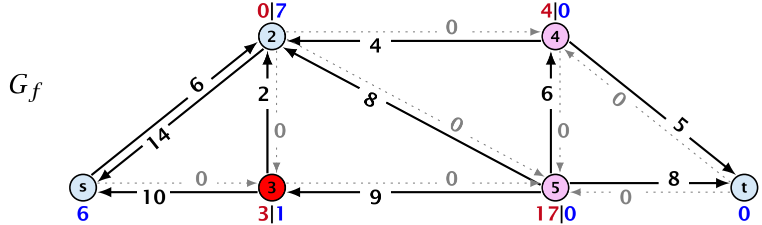
Preflow Push Algorithm



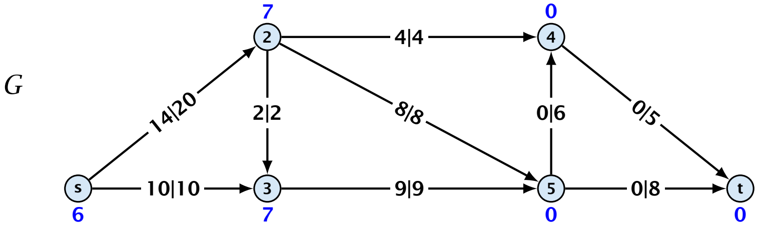
Preflow Push Algorithm

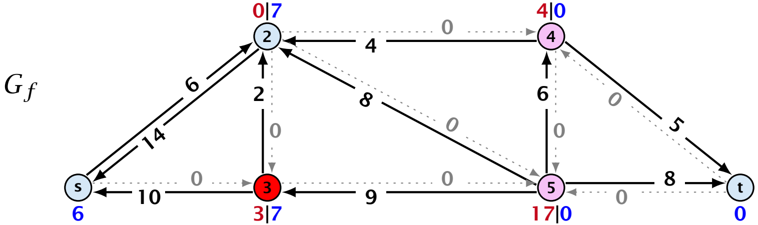
relabel 6 times





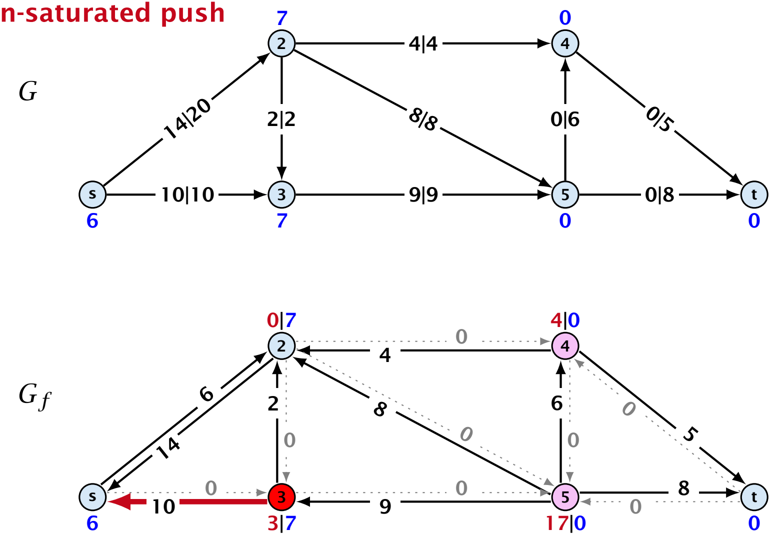
Preflow Push Algorithm





Preflow Push Algorithm

non-saturated push



Preflow Push Algorithm

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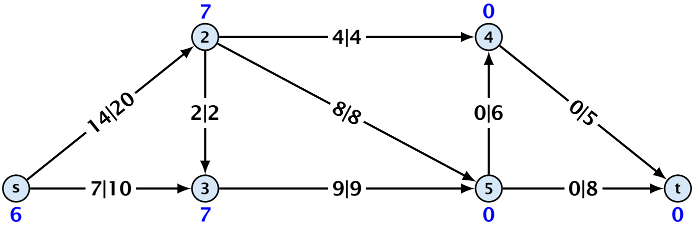
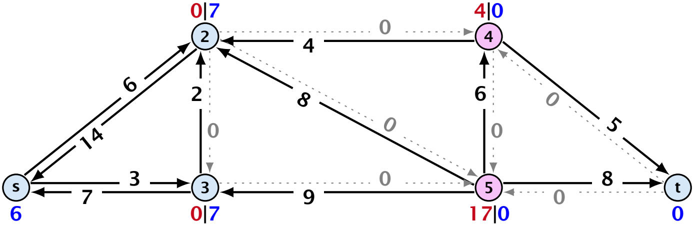


G_f



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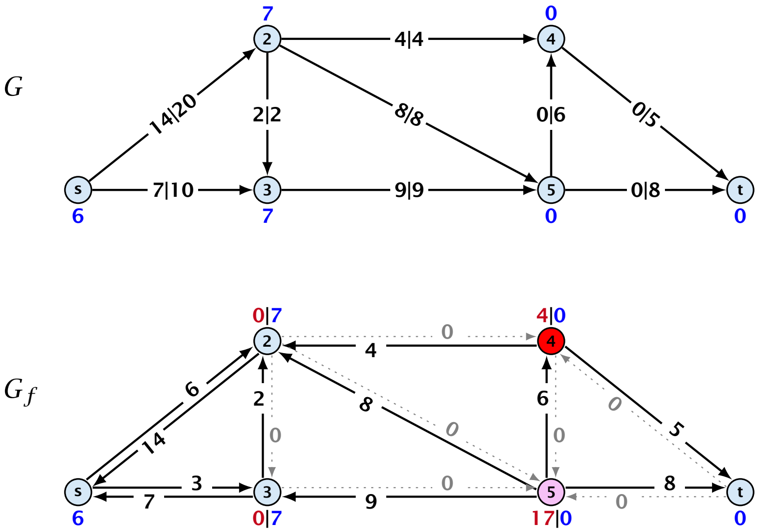
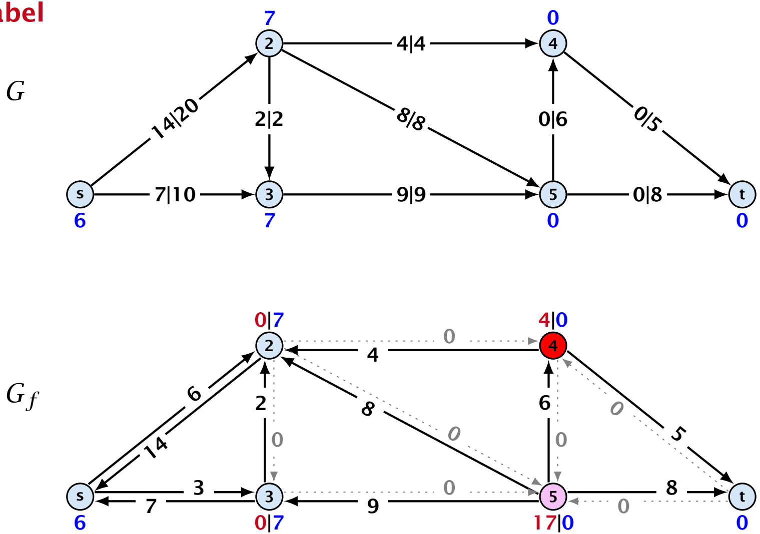


Preflow Push Algorithm

relabel



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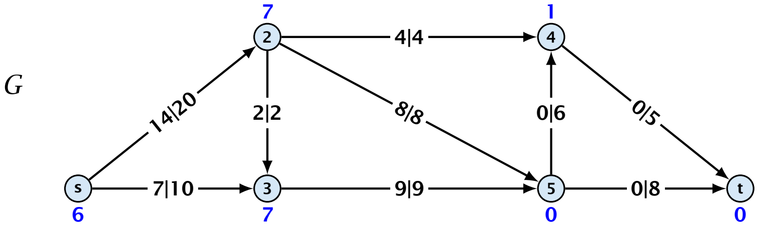
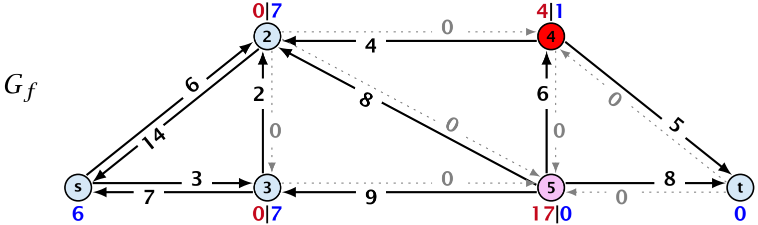
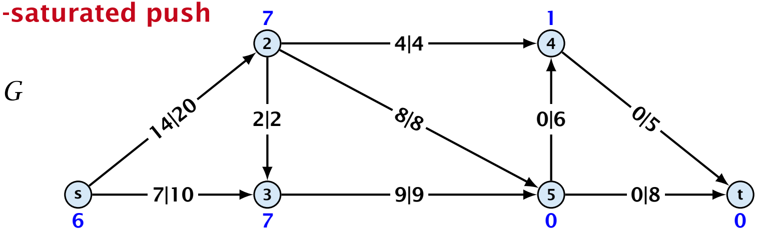
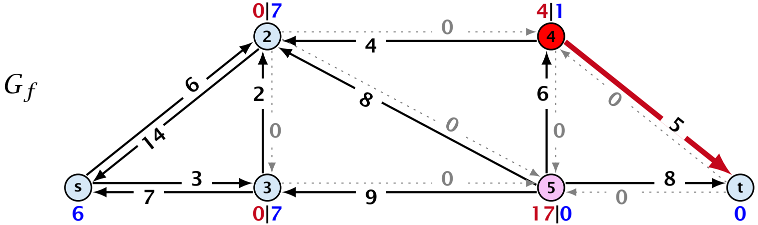




Preflow Push Algorithm

non-saturated push





Preflow Push Algorithm

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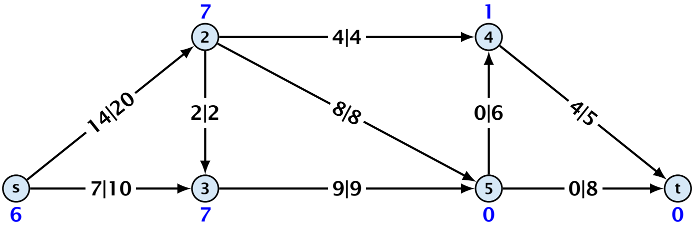

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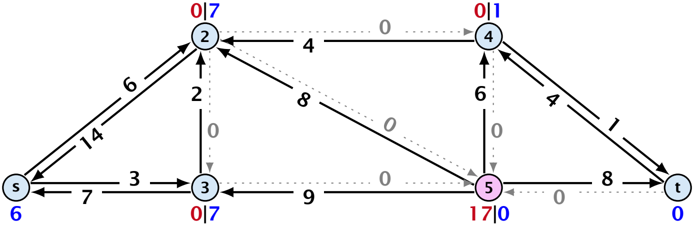
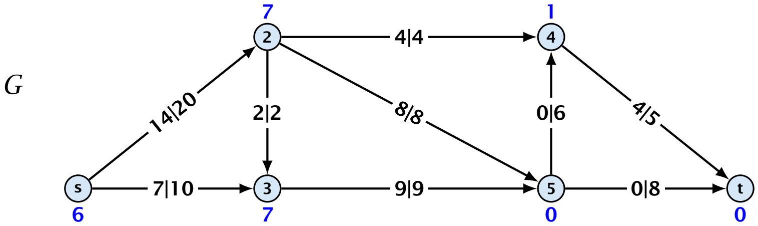
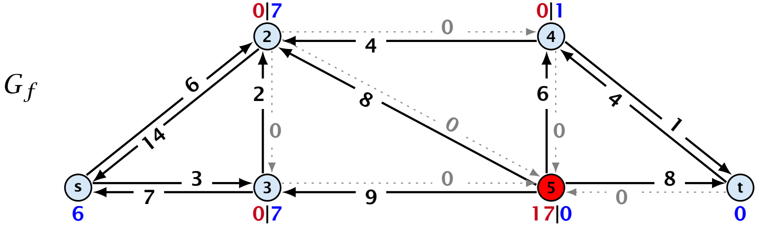


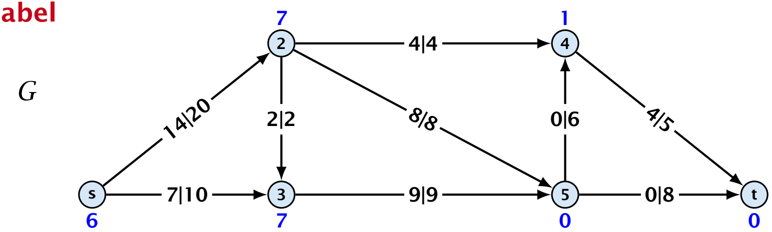
Preflow Push Algorithm

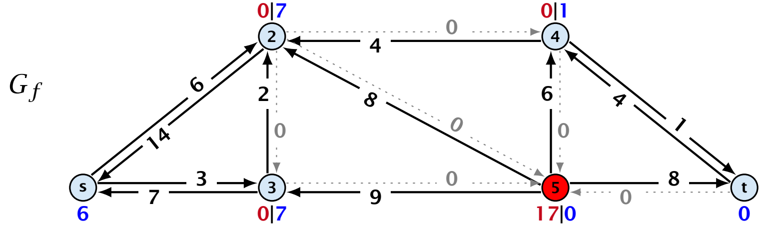




Preflow Push Algorithm

relabel





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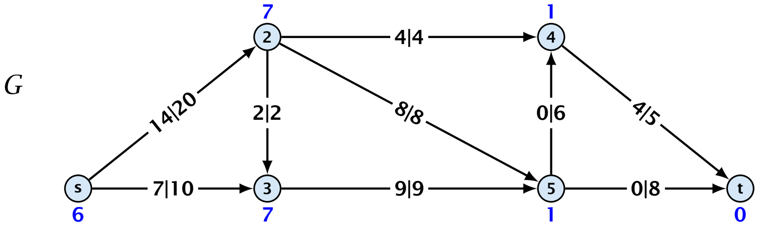
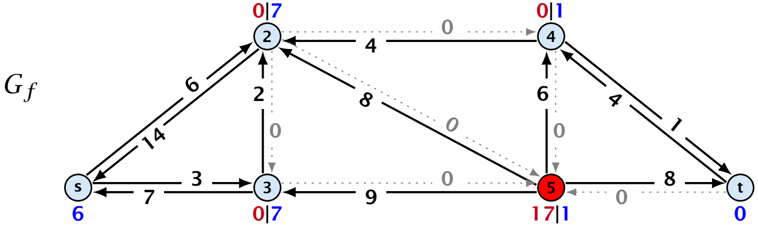
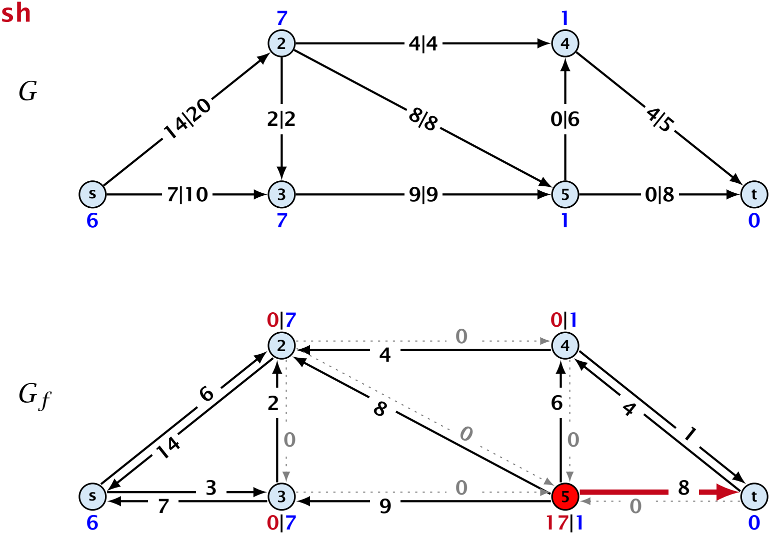


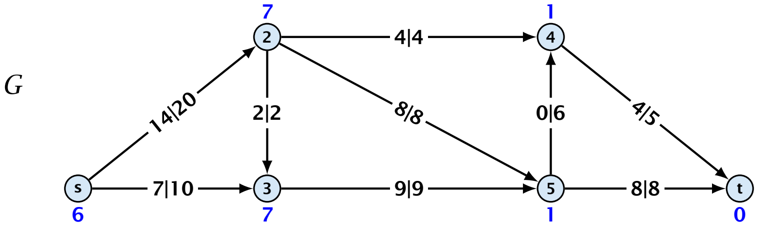


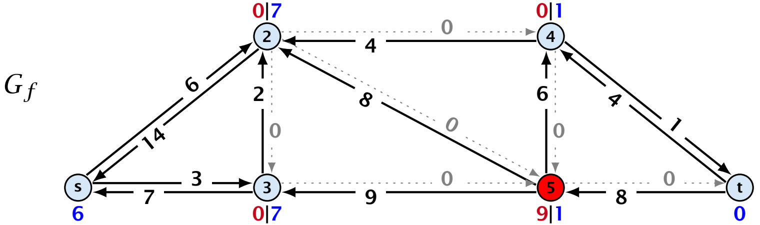
Preflow Push Algorithm

push



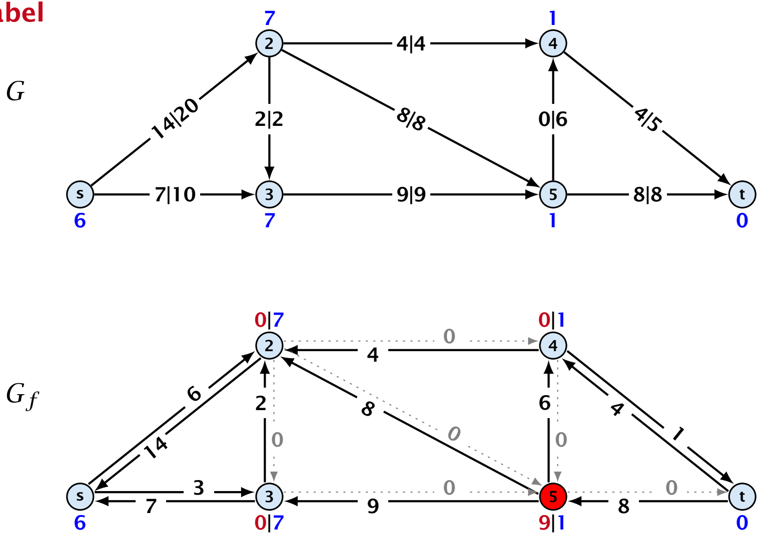
Preflow Push Algorithm





Preflow Push Algorithm

relabel



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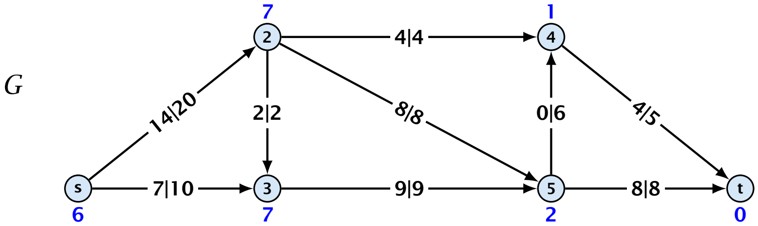
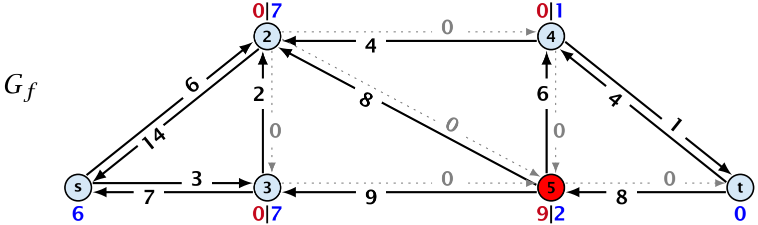
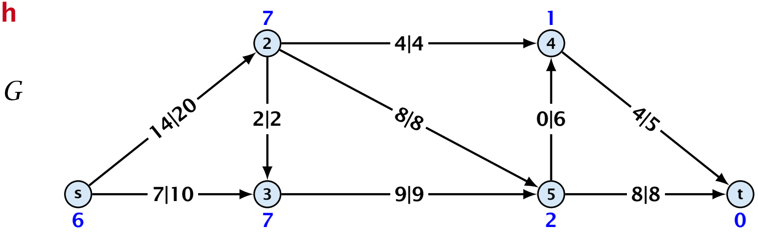
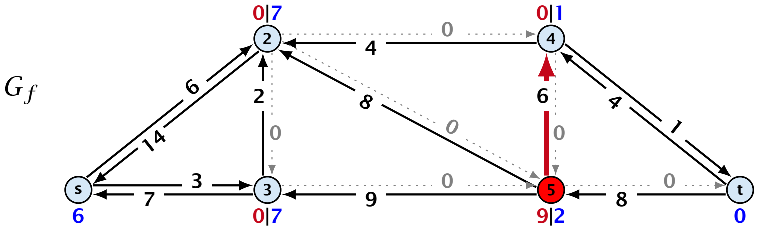




Preflow Push Algorithm

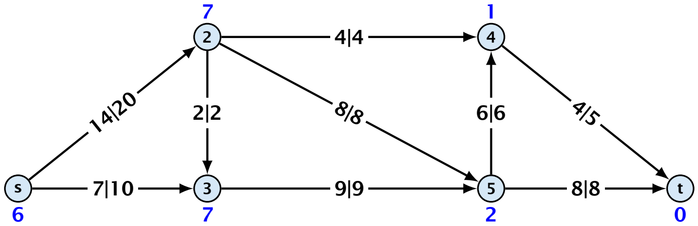
push



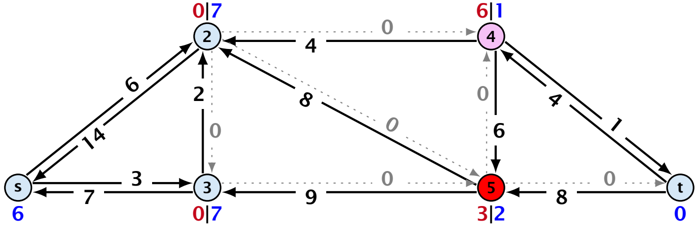


Preflow Push Algorithm

G



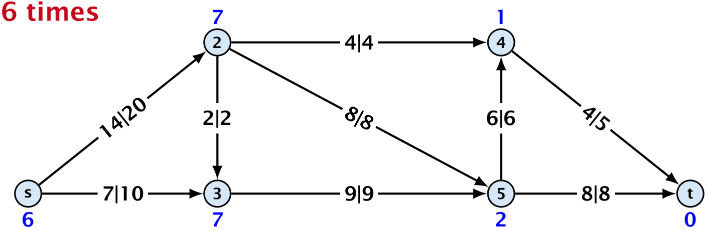
G_f



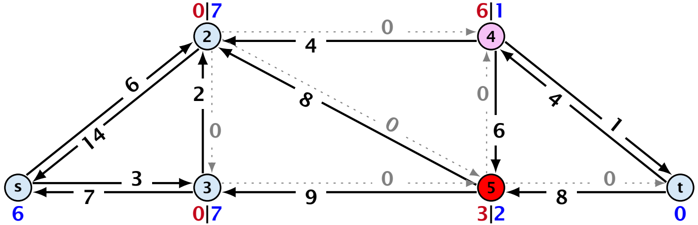
Preflow Push Algorithm

relabel 6 times

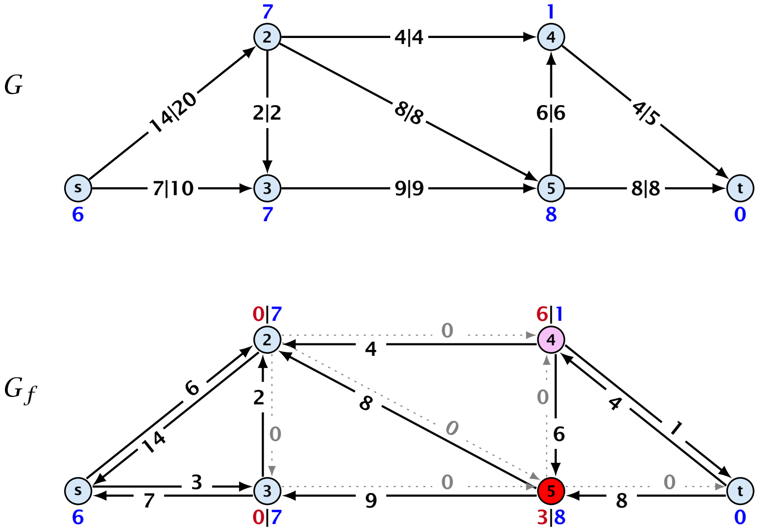
G



G_f

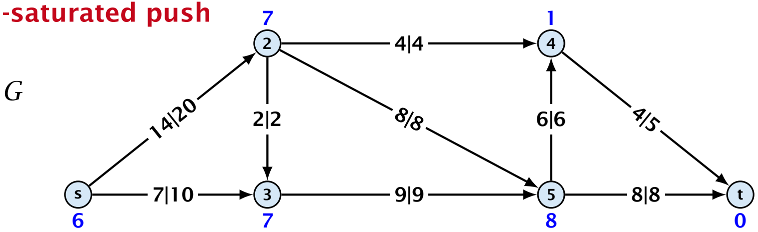


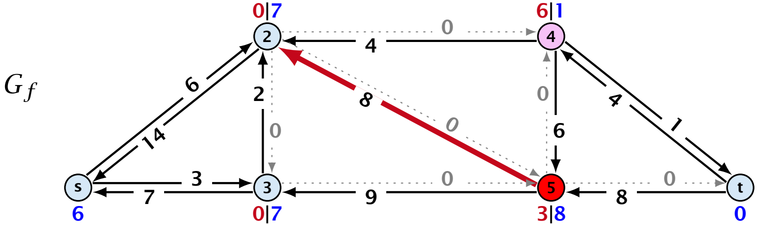
Preflow Push Algorithm



Preflow Push Algorithm

non-saturated push





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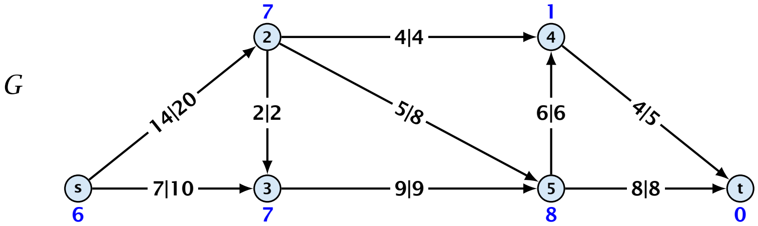

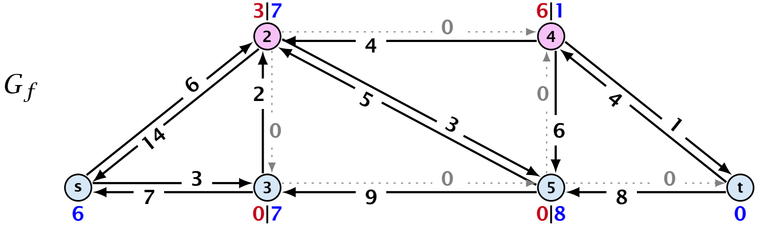

Preflow Push Algorithm

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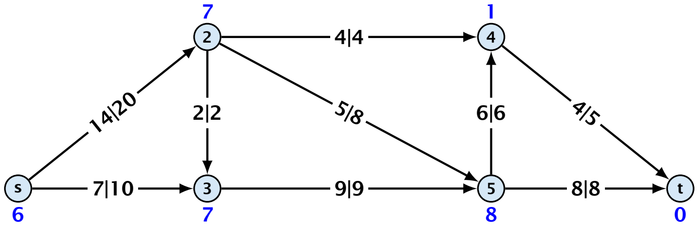

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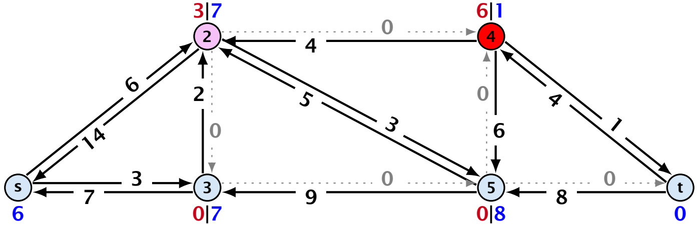
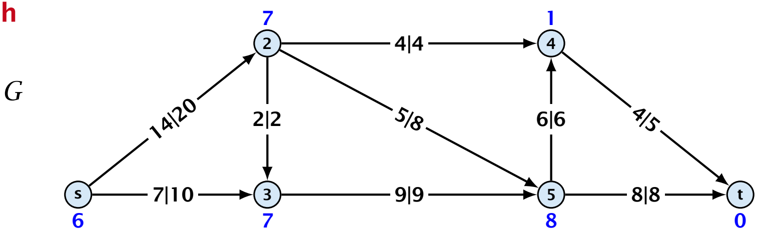
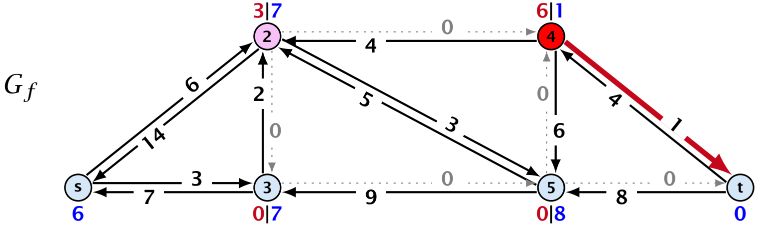


Preflow Push Algorithm

push





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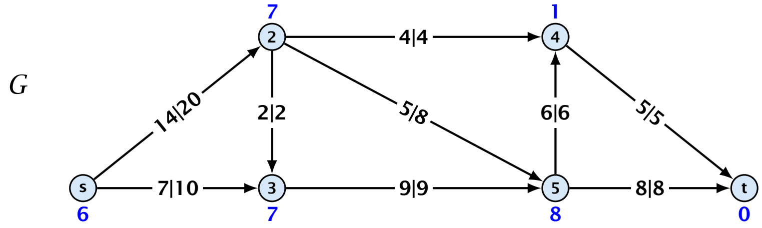
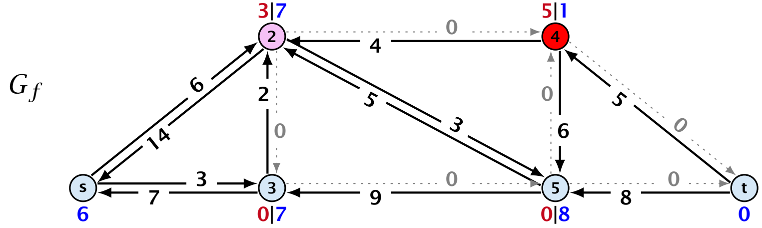
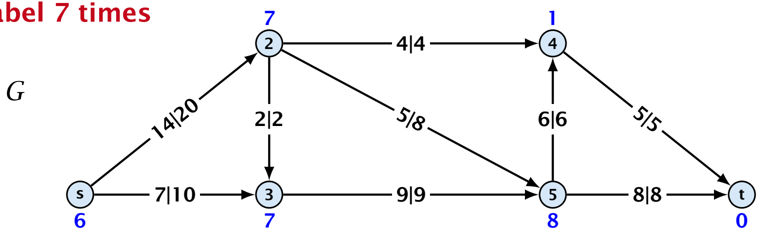
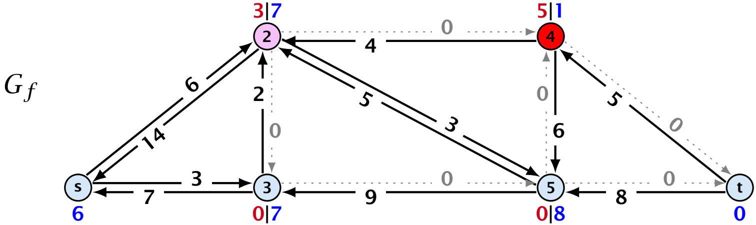




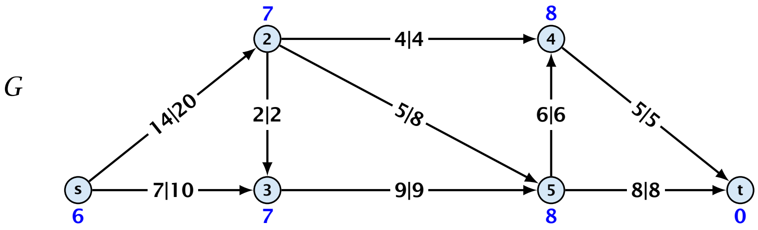
Preflow Push Algorithm

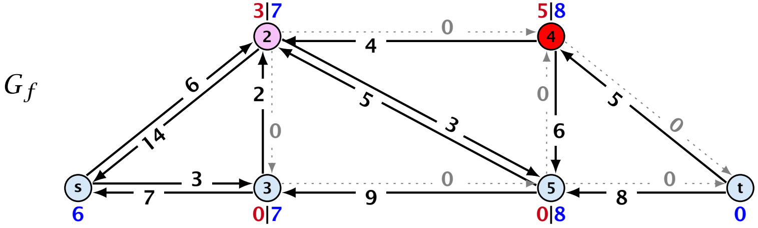
relabel 7 times





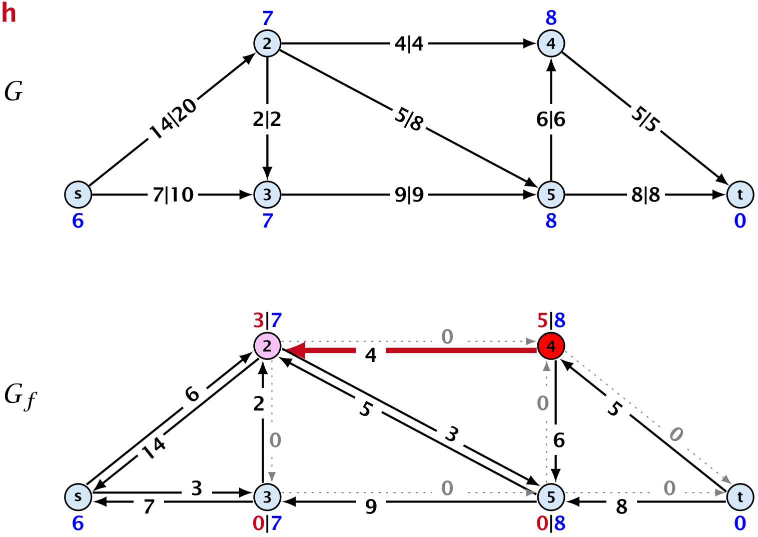
Preflow Push Algorithm



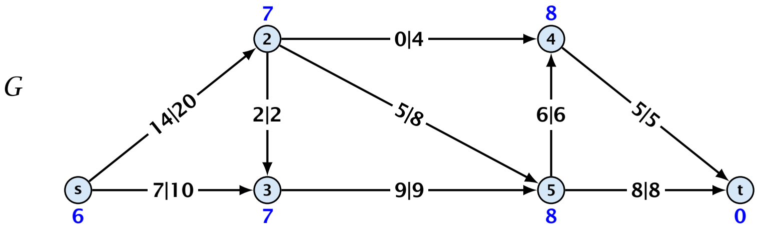


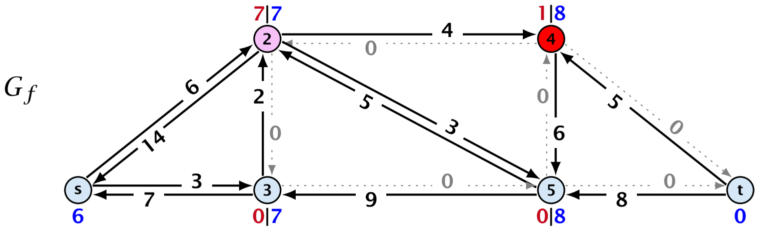
Preflow Push Algorithm

push



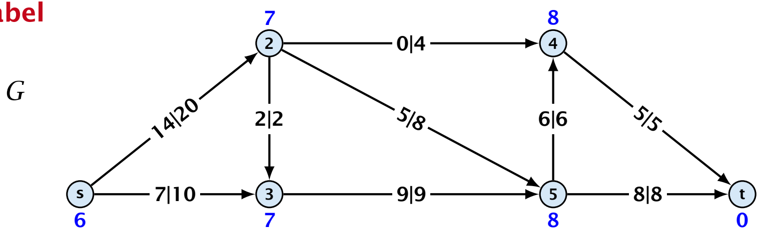
Preflow Push Algorithm

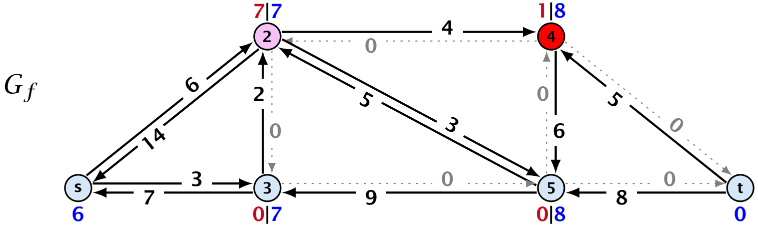




Preflow Push Algorithm

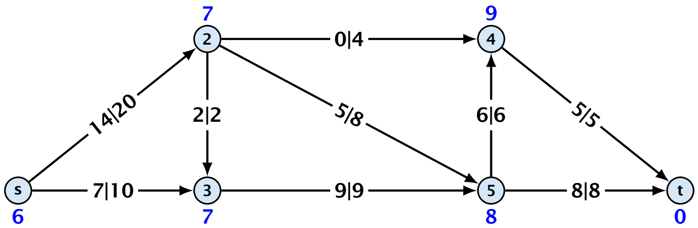
relabel



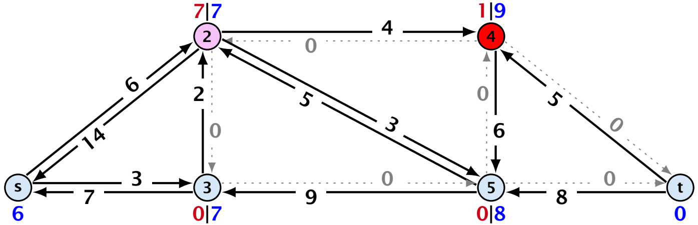


Preflow Push Algorithm

G

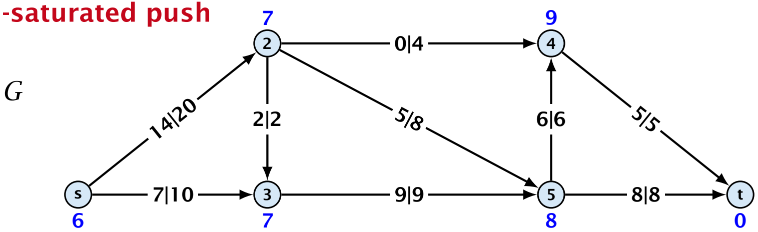


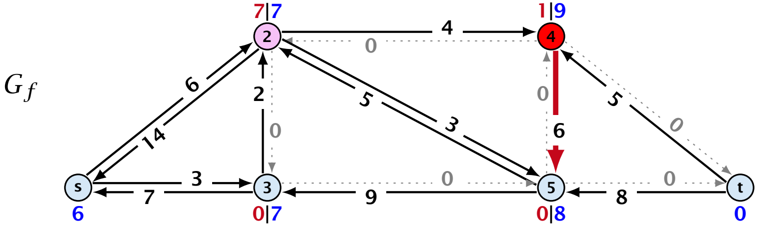
G_f



Preflow Push Algorithm

non-saturated push





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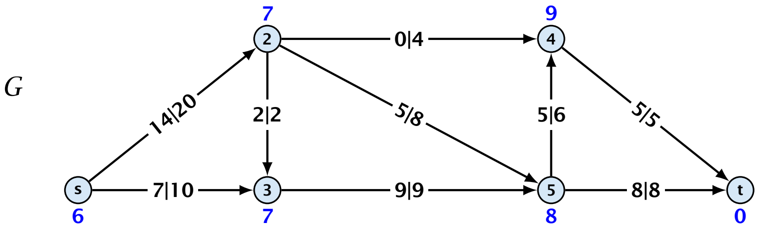

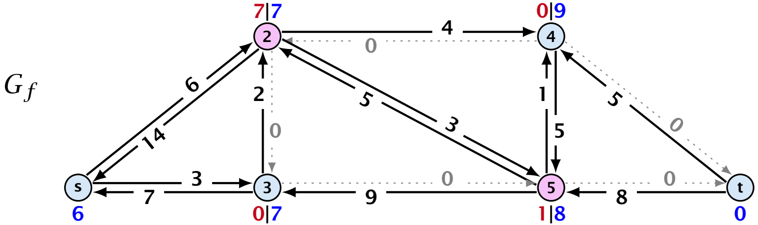

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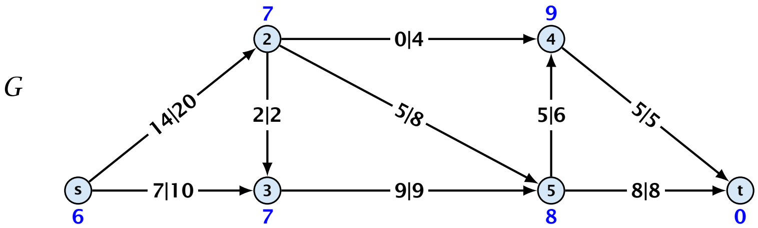
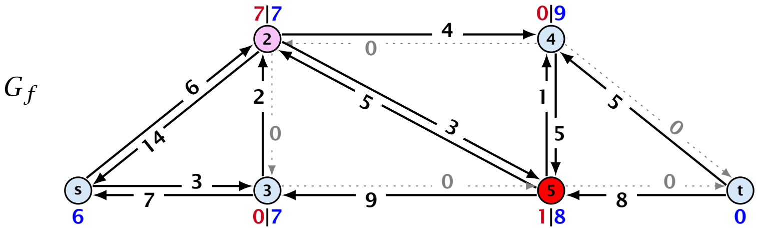
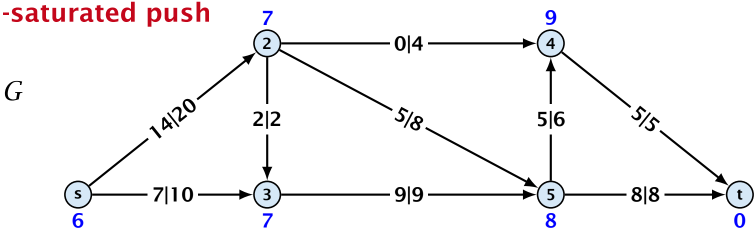
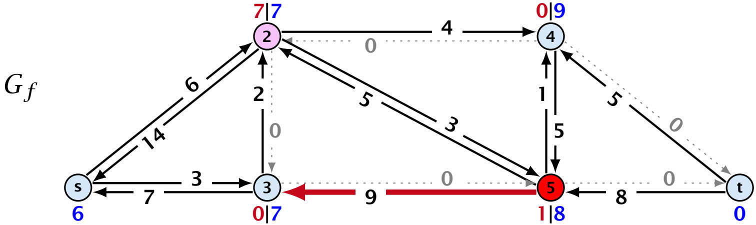




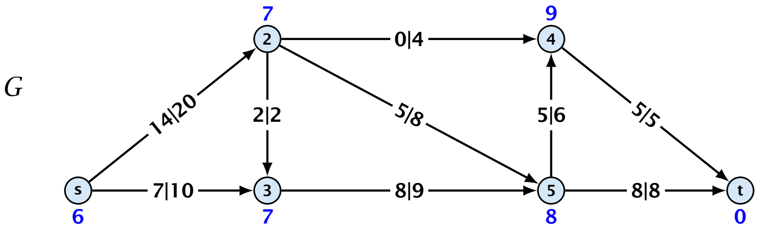
Preflow Push Algorithm

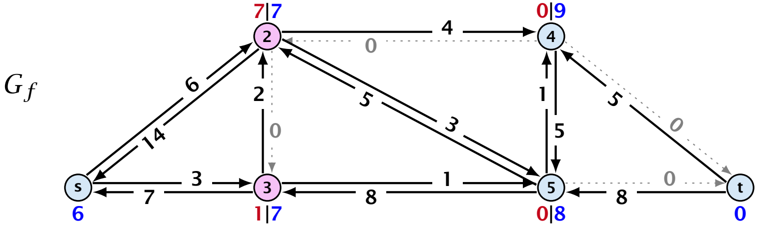
non-saturated push



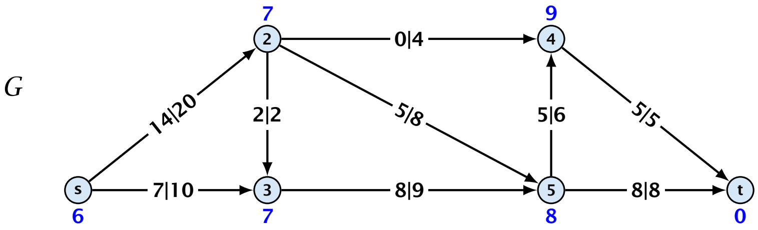


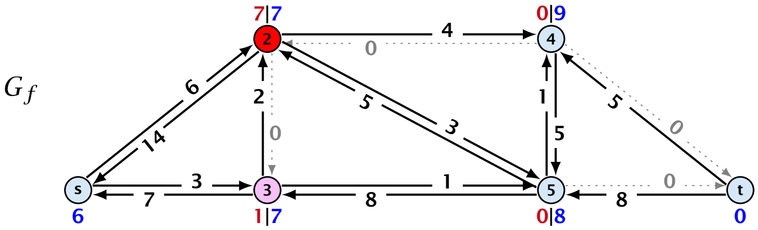
Preflow Push Algorithm





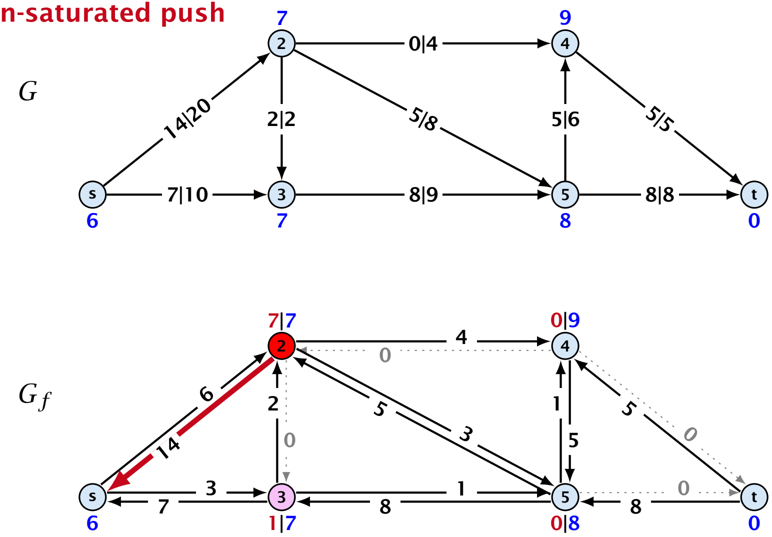
Preflow Push Algorithm



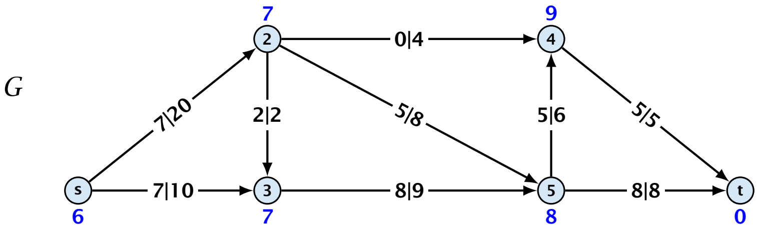


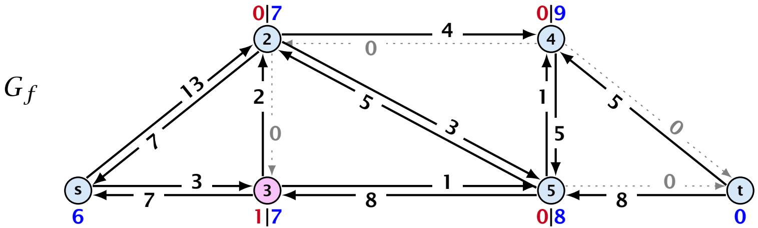
Preflow Push Algorithm

non-saturated push

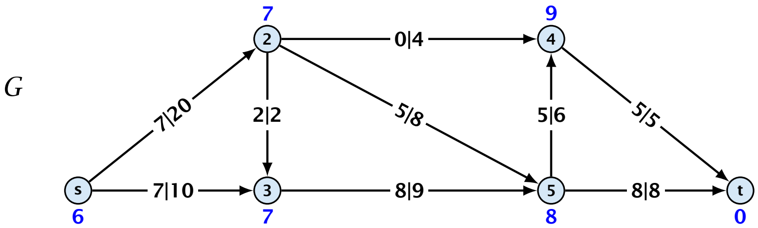


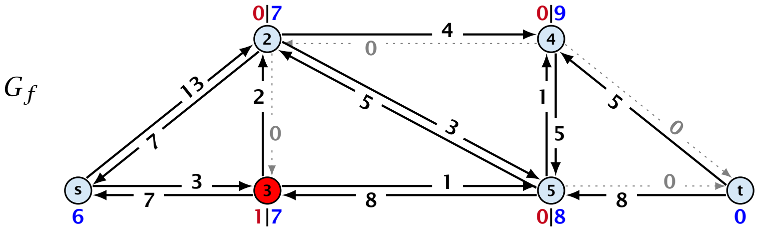
Preflow Push Algorithm





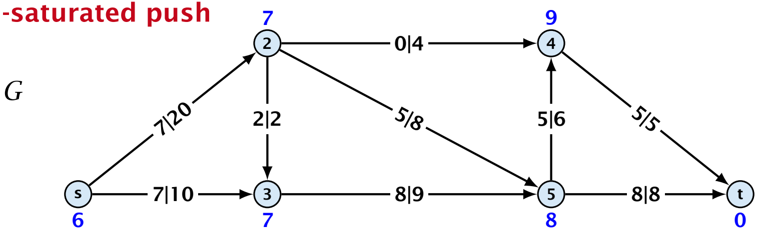
Preflow Push Algorithm

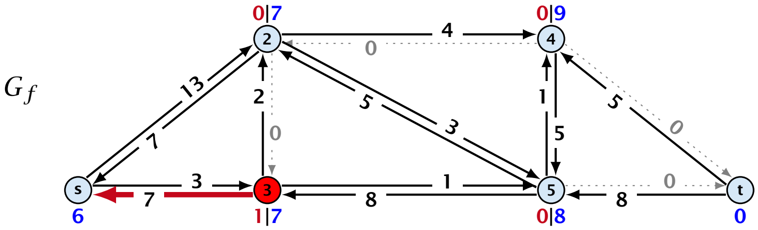




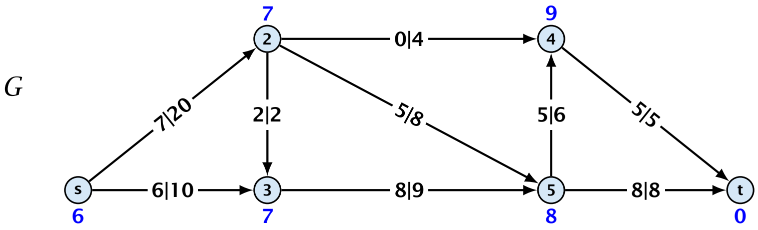
Preflow Push Algorithm

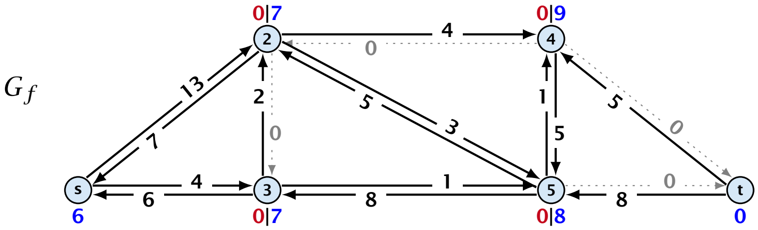
non-saturated push





Preflow Push Algorithm





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- In the following we show that a node $b \in B$ has excess flow $f(b) = 0$ which gives the lemma.
- In the residual graph there are no edges into A, and, hence, no edges leaving A/entering B can carry any flow.
- Let $f(B) = \sum_{v \in B} f(v)$ be the excess flow of all nodes in B.

Let $f : E \to \mathbb{R}_0^+$ be a preflow. We introduce the notation

$$f(x,y) = \begin{cases} 0 & (x,y) \notin E \\ f((x,y)) & (x,y) \in E \end{cases}$$

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Hence, the excess flow $f(b)$ must be 0 for every node $b \in B$.

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There are only $\mathcal{O}(n^2)$ relabel operations.

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- Since the label of v is at most $2n - 1$, there are at most n pushes along (u, v).

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- Hence,

$$\begin{aligned}\text{\#non-saturating\_pushes} &\leq \text{\#relabels} + 2n \cdot \text{\#saturating\_pushes} \\ &\leq \mathcal{O}(n^2m) \quad .\end{aligned}$$

Analysis

Theorem 10

There is an implementation of the generic push relabel algorithm with running time $\mathcal{O}(n^2m)$.

For every node maintain a list of admissable edges starting at that node. Further maintain a list of active nodes.

A push along an edge (u,v) can be performed in constant time

- check whether edge (v,u) needs to be added to G_f
- check whether (u,v) needs to be deleted (saturating push)
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A relabel at a node u can be performed in time $\mathcal{O}(n)$

- check for all outgoing edges if they become admissable
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For special variants of push relabel algorithms we organize the neighbours of a node into a linked list (possible neighbours in the residual graph G_f). Then we use the discharge-operation:

Algorithm 47 discharge(u)

```
while u is active do
    v ← u.current-neighbour
    if v = null then
        relabel(u)
        u.current-neighbour ← u.neighbour-list-head
    else
        if (u, v) admissable then push(u, v)
        else u.current-neighbour ← v.next-in-list
```

Note that u.*current-neighbour* is a global variable. It is only changed within the discharge routine, but keeps its value between consecutive calls to discharge.

Lemma 11
If $v = \text{null}$ *in Line 3, then there is no outgoing admissable edge from* u.

Proof.

- While pushing from u the current-neighbour pointer is only advanced if the current edge is not admissable.
- The only thing that could make the edge admissable again would be a relabel at u.
- If we reach the end of the list ($v = \text{null}$) all edges are not admissable. □

This shows that discharge(u) is correct, and that we can perform a relabel in line 4.

13.2 Relabel to Front

Algorithm 48 relabel-to-front(G, s, t)

```
initialize preflow
initialize node list L containing V \ {s, t} in any order
foreach u ∈ V \ {s, t} do
    u.current-neighbour ← u.neighbour-list-head
u ← L.head
while u ≠ null do
    old-height ← ℓ(u)
    discharge(u)
    if ℓ(u) > old-height then // relabel happened
        move u to the front of L
    u ← u.next
```

13.2 Relabel to Front

Lemma 12 (Invariant)

In Line 6 of the relabel-to-front algorithm the following invariant holds.

1. *The sequence L is topologically sorted w.r.t. the set of admissable edges; this means for an admissable edge (x, y) the node x appears before y in sequence L.*
2. *No node before u in the list L is active.*

Proof:

- Initialization:
  1. In the beginning s has label $n \geq 2$, and all other nodes have label 0. Hence, no edge is admissable, which means that any ordering L is permitted.
  2. We start with u being the head of the list; hence no node before u can be active
- Maintenance:
  1.
     - Pushes do no create any new admissable edges. Therefore, if discharge$()$ does not relabel u, L is still topologically sorted.
     - After relabeling, u cannot have admissable incoming edges as such an edge (x, u) would have had a difference $\ell(x) - \ell(u) \geq 2$ before the re-labeling (such edges do not exist in the residual graph).
       Hence, moving u to the front does not violate the sorting property for any edge; however it fixes this property for all admissable edges leaving u that were generated by the relabeling.

13.2 Relabel to Front

Proof:

- Maintenance:
  2. If we do a relabel there is nothing to prove because the only node before u' (u in the next iteration) will be the current u; the discharge(u) operation only terminates when u is not active anymore.

     For the case that we do not relabel, observe that the only way a predecessor could be active is that we push flow to it via an admissable arc. However, all admissable arc point to successors of u.

Note that the invariant means that for $u = \text{null}$ we have a preflow with a valid labelling that does not have active nodes. This means we have a maximum flow.

13.2 Relabel to Front

Lemma 13
There are at most $\mathcal{O}(n^3)$ calls to discharge(u).

Every discharge operation without a relabel advances u (the current node within list L). Hence, if we have n discharge operations without a relabel we have $u = \mathrm{null}$ and the algorithm terminates.

Therefore, the number of calls to discharge is at most $n(\#relabels + 1) = \mathcal{O}(n^3)$.

13.2 Relabel to Front

Lemma 14
The cost for all relabel-operations is only $\mathcal{O}(n^2)$.

A relabel-operation at a node is constant time (increasing the label and resetting $u.current\text{-}neighbour$). In total we have $\mathcal{O}(n^2)$ relabel-operations.

13.2 Relabel to Front

Note that by definition a saturing push operation $(\min\{c_f(e), f(u)\} = c_f(e))$ can at the same time be a non-saturating push operation $(\min\{c_f(e), f(u)\} = f(u))$.

Lemma 15

*The cost for all saturating push-operations that are **not** also non-saturating push-operations is only $\mathcal{O}(mn)$.*

Note that such a push-operation leaves the node u active but makes the edge e disappear from the residual graph. Therefore the push-operation is immediately followed by an increase of the pointer $u.\mathit{current\text{-}neighbour}$.

This pointer can traverse the neighbour-list at most $\mathcal{O}(n)$ times (upper bound on number of relabels) and the neighbour-list has only $\mathit{degree}(u) + 1$ many entries ($+1$ for null-entry).

13.2 Relabel to Front

Lemma 16
The cost for all non-saturating push-operations is only $\mathcal{O}(n^3)$.

A non-saturating push-operation takes constant time and ends the current call to discharge(). Hence, there are only $\mathcal{O}(n^3)$ such operations.

Theorem 17
The push-relabel algorithm with the rule relabel-to-front takes time $\mathcal{O}(n^3)$.

13.3 Highest label

Algorithm 49 highest-label(G, s, t)

1: initialize preflow
2: **foreach** $u \in V \setminus \{s, t\}$ **do**
3: $\quad u.current\text{-}neighbour \leftarrow u.neighbour\text{-}list\text{-}head$
4: **while** $\exists$ active node u **do**
5: $\quad$ select active node u with highest label
6: $\quad$ discharge(u)

13.3 Highest label

Lemma 18
When using highest label the number of non-saturating pushes is only $\mathcal{O}(n^3)$.

A push from a node on level ℓ can only "activate" nodes on levels strictly less than ℓ.

This means, after a non-saturating push from u a relabel is required to make u active again.

Hence, after n non-saturating pushes without an intermediate relabel there are no active nodes left.

Therefore, the number of non-saturating pushes is at most $n(\#relabels + 1) = \mathcal{O}(n^3)$.

13.3 Highest label

Since a discharge-operation is terminated by a non-saturating push this gives an upper bound of $\mathcal{O}(n^3)$ on the number of discharge-operations.

The cost for relabels and saturating pushes can be estimated in exactly the same way as in the case of the generic push-relabel algorithm.

Question:
How do we find the next node for a discharge operation?

13.3 Highest label

Maintain lists L_i, $i \in \{0, \dots, 2n\}$, where list L_i contains active nodes with label i (maintaining these lists induces only constant additional cost for every push-operation and for every relabel-operation).

After a discharge operation terminated for a node u with label k, traverse the lists $L_k, L_{k-1}, \dots, L_0$, (in that order) until you find a non-empty list.

Unless the last (non-saturating) push was to s or t the list $k - 1$ must be non-empty (i.e., the search takes constant time).

13.3 Highest label

Hence, the total time required for searching for active nodes is at most

$$\mathcal{O}(n^3) + n(\textit{\#non-saturating-pushes-to-s-or-t})$$

Lemma 19

The number of non-saturating pushes to s or t is at most $\mathcal{O}(n^2)$.

With this lemma we get

Theorem 20

The push-relabel algorithm with the rule highest-label takes time $\mathcal{O}(n^3)$.

13.3 Highest label

Proof of the Lemma.

- We only show that the number of pushes to the source is at most $\mathcal{O}(n^2)$. A similar argument holds for the target.
- After a node v (which must have $\ell(v) = n + 1$) made a non-saturating push to the source there needs to be another node whose label is increased from $\le n + 1$ to $n + 2$ before v can become active again.
- This happens for every push that v makes to the source. Since, every node can pass the threshold $n + 2$ at most once, v can make at most n pushes to the source.
- As this holds for every node the total number of pushes to the source is at most $\mathcal{O}(n^2)$.