Duality

How do we get an upper bound to a maximization LP?

$$\begin{array}{rrcrl} \text{max} & 13a & + & 23b & \\ \text{s.t.} & 5a & + & 15b & \leq 480 \\ & 4a & + & 4b & \leq 160 \\ & 35a & + & 20b & \leq 1190 \\ & & & a, b & \geq 0 \end{array}$$

Note that a lower bound is easy to derive. Every choice of $a, b \geq 0$ gives us a lower bound (e.g. $a = 12, b = 28$ gives us a lower bound of 800).

If you take a conic combination of the rows (multiply the i-th row with $y_i \geq 0$) such that $\sum_i y_i a_{ij} \geq c_j$ then $\sum_i y_i b_i$ will be an upper bound.

Duality

Definition 2
Let $z = \max\{c^t x \mid Ax \geq b, x \geq 0\}$ be a linear program P (called the primal linear program).

The linear program D defined by

$$w = \min\{b^t y \mid A^t y \geq c, y \geq 0\}$$

is called the dual problem.

Duality

Lemma 3

The dual of the dual problem is the primal problem.

Proof:

- $w = \min\{b^t y \mid A^t y \geq c, y \geq 0\}$
- $w = \max\{-b^t y \mid -A^t y \leq -c, y \geq 0\}$

The dual problem is

- $z = \min\{-c^t x \mid -Ax \geq -b, x \geq 0\}$
- $z = \max\{c^t x \mid Ax \geq b, x \geq 0\}$

Weak Duality

Let $z = \max\{c^t x \mid Ax \leq b, x \geq 0\}$ and
$w = \min\{b^t y \mid A^t y \geq c, y \geq 0\}$ be a primal dual pair.

x is primal feasible iff $x \in \{x \mid Ax \leq b, x \geq 0\}$

y is dual feasible, iff $y \in \{y \mid A^t y \geq c, y \geq 0\}$.

Theorem 4 (Weak Duality)

Let $\hat{x}$ be primal feasible and let $\hat{y}$ be dual feasible. Then

$$c^t \hat{x} \leq z \leq w \leq b^t \hat{y} \ .$$

Weak Duality

$A^t\hat{y} \geq c \Rightarrow \hat{x}^tA^t\hat{y} \geq \hat{x}^tc$ $(\hat{x} \geq 0)$

$A\hat{x} \leq b \Rightarrow y^tA\hat{x} \leq \hat{y}^tb$ $(\hat{y} \geq 0)$

This gives

$$c^t\hat{x} \leq \hat{y}^tA\hat{x} \leq b^t\hat{y} \ .$$

Since, there exists primal feasible $\hat{x}$ with $c^t\hat{x} = z$, and dual feasible $\hat{y}$ with $b^ty = w$ we get $z \leq w$.

If P is unbounded then D is infeasible.

The following linear programs form a primal dual pair:

$$z = \max\{c^t x \mid Ax = b, x \geq 0\}$$
$$w = \min\{b^t y \mid A^t y \geq c\}$$

This means for computing the dual of a standard form LP, we do not have non-negativity constraints for the dual variables.

Proof

Primal:

$$\max\{c^t x \mid Ax = b, x \geq 0\}$$
$$= \max\{c^t x \mid Ax \leq b, -Ax \leq -b, x \geq 0\}$$
$$= \max\{c^t x \mid \begin{bmatrix} A \\ -A \end{bmatrix} x \leq \begin{bmatrix} b \\ -b \end{bmatrix}, x \geq 0\}$$

Dual:

$$\min\{[b^t \; -b^t] y \mid [A^t \; -A^t] y \geq c, y \geq 0\}$$
$$= \min\left\{[b^t \; -b^t] \cdot \begin{bmatrix} y^+ \\ y^- \end{bmatrix} \;\middle|\; [A^t \; -A^t] \cdot \begin{bmatrix} y^+ \\ y^- \end{bmatrix} \geq c, y^- \geq 0, y^+ \geq 0\right\}$$
$$= \min\left\{b^t \cdot (y^+ - y^-) \;\middle|\; A^t \cdot (y^+ - y^-) \geq c, y^- \geq 0, y^+ \geq 0\right\}$$
$$= \min\left\{b^t y' \;\middle|\; A^t y' \geq c, y' \geq 0\right\}$$

Proof of Optimality Criterion for Simplex

Suppose that we have a basic feasible solution with reduced cost

$$\tilde{c} = c^t - c_B^t A_B^{-1} A \le 0$$

This is equivalent to $A^t (A_B^{-1})^t c_B \ge c$

$y^* = (A_B^{-1})^t c_B$ is solution to the dual $\min\{b^t y | A^t y \ge c\}$.

$$\begin{aligned} b^t y^* &= (Ax^*)^t y^* = (A_B x_B^*)^t y^* \\ &= (A_B x_B^*)^t (A_B^{-1})^t c_B = (x_B^*)^t A_B^t (A_B^{-1})^t c_B \\ &= c^t x^* \end{aligned}$$

Hence, the solution is optimal.

Strong Duality

Theorem 5 (Strong Duality)

Let P and D be a primal dual pair of linear programs, and let z^* and w^* denote the optimal solution to P and D, respectively. Then

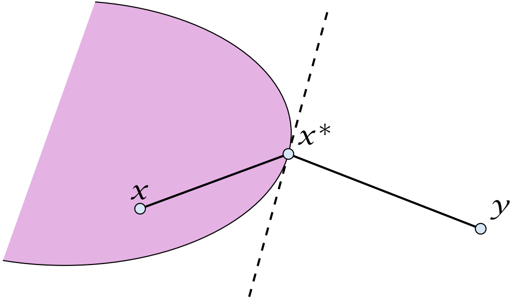
$$z^* = w^*$$

Lemma 6 (Weierstrass)

Let X be a compact set and let $f(x)$ be a continuous function on X. Then $\min\{f(x) : x \in X\}$ exists.

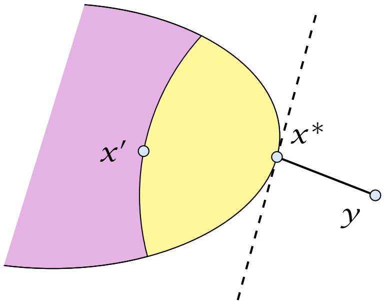
Lemma 7 (Projection Lemma)

Let $X \subseteq \mathbb{R}^m$ be a non-empty convex set, and let $y \notin X$. Then there exist $x^* \in X$ with minimum distance from y. Moreover for all $x \in X$ we have $(y - x^*)^t(x - x^*) \leq 0$.



Proof of the Projection Lemma

- Define $f(x) = \|y - x\|$.
- We want to apply Weierstrass but X may not be bounded.
- $X \neq \emptyset$. Hence, there exists $x' \in X$.
- Define $X' = \{x \in X \mid \|y - x\| \leq \|y - x'\|\}$. This set is closed and bounded.
- Applying Weierstrass gives the existence.



Proof of the Projection Lemma (continued)

x^* is minimum. Hence $\|y - x^*\|^2 \leq \|y - x\|^2$ for all $x \in X$.

By convexity: $x \in X$ then $x^* + \epsilon(x - x^*) \in X$ for all $0 \leq \epsilon \leq 1$.

$$\begin{aligned}\|y - x^*\|^2 &\leq \|y - x^* - \epsilon(x - x^*)\|^2 \\ &= \|y - x^*\|^2 + \epsilon^2\|x - x^*\|^2 - 2\epsilon(y - x^*)^t(x - x^*)\end{aligned}$$

Hence, $(y - x^*)^t(x - x^*) \leq \frac{1}{2}\epsilon\|x - x^*\|^2$.

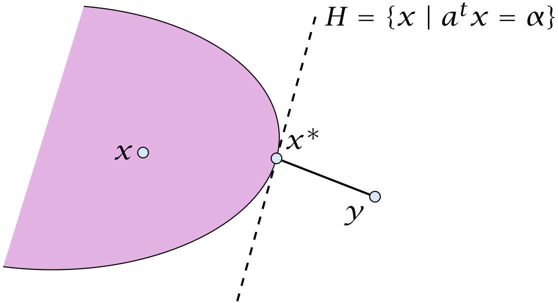
Letting $\epsilon \to 0$ gives the result.

Theorem 8 (Separating Hyperplane)

Let $X \subseteq \mathbb{R}^m$ be a non-empty closed convex set, and let $y \notin X$. Then there exists a separating hyperplane $\{x \in \mathbb{R} : a^t x = \alpha\}$ where $a \in \mathbb{R}^m$, $\alpha \in \mathbb{R}$ that separates y from X. ($a^t y < \alpha$; $a^t x \geq \alpha$ for all $x \in X$)

Proof of the Hyperplane Lemma

- Let $x^* \in X$ be closest point to y in X.
- By previous lemma $(y - x^*)^t(x - x^*) \leq 0$ for all $x \in X$.
- Choose $a = (x^* - y)$ and $\alpha = a^t x^*$.
- For $x \in X$: $a^t(x - x^*) \geq 0$, and, hence, $a^t x \geq \alpha$.
- Also, $a^t y = a^t(x^* - a) = \alpha - \|a\|^2 < \alpha$



Lemma 9 (Farkas Lemma)

Let A be an $m \times n$ matrix, $b \in \mathbb{R}^m$. Then exactly one of the following statements holds.

1. $\exists x \in \mathbb{R}^n$ *with* $Ax = b$, $x \geq 0$
2. $\exists y \in \mathbb{R}^m$ *with* $A^t y \geq 0$, $b^t y < 0$

Assume $\hat{x}$ satisfies 1. and $\hat{y}$ satisfies 2. Then

$$0 > y^t b = y^t Ax \geq 0$$

Hence, at most one of the statements can hold.

Proof of Farkas Lemma

Now, assume that 1. does not hold.

Consider $S = \{Ax : x \geq 0\}$ so that S closed, convex, $b \notin S$.

We want to show that there is y with $A^t y \geq 0$, $b^t y < 0$.

Let y be a hyperplane that separates b from S. Hence, $y^t b < \alpha$ and $y^t s \geq \alpha$ for all $s \in S$.

$0 \in S \Rightarrow \alpha \leq 0 \Rightarrow y^t b < 0$

$y^t Ax \geq \alpha$ for all $x \geq 0$. Hence, $y^t A \geq 0$ as we can choose x arbitrarily large.

Lemma 10 (Farkas Lemma; different version)

Let A be an $m \times n$ matrix, $b \in \mathbb{R}^m$. Then exactly one of the following statements holds.

1. $\exists x \in \mathbb{R}^n$ *with* $Ax \le b$, $x \ge 0$
2. $\exists y \in \mathbb{R}^m$ *with* $A^t y \ge 0$, $b^t y < 0$, $y \ge 0$

Rewrite the conditions:

1. $\exists x \in \mathbb{R}^n$ with $\begin{bmatrix} A & I \end{bmatrix} \cdot \begin{bmatrix} x \\ s \end{bmatrix} = b$, $x \ge 0$, $s \ge 0$
2. $\exists y \in \mathbb{R}^m$ with $\begin{bmatrix} A^t \\ I \end{bmatrix} y \ge 0$, $b^t y < 0$

Proof of Strong Duality

$$P\colon\ z = \max\{c^t x \mid Ax \le b, x \ge 0\}$$

$$D\colon\ w = \min\{b^t y \mid A^t y \ge c, y \ge 0\}$$

Theorem 11 (Strong Duality)

Let P and D be a primal dual pair of linear programs, and let z and w denote the optimal solution to P and D, respectively (i.e., P and D are non-empty). Then

$$z = w\ .$$

Proof of Strong Duality

$\boldsymbol{z \leq w}$: follows from weak duality

$\boldsymbol{z \geq w}$:
We show $z < \alpha$ implies $w < \alpha$.

$$\begin{array}{l}\exists x \in \mathbb{R}^n \\ \begin{array}{rrcl} \text{s.t.} & Ax & \leq & b \\ & -c^t x & \leq & -\alpha \\ & x & \geq & 0 \end{array}\end{array}$$

$$\begin{array}{l}\exists y \in \mathbb{R}^m; z \in \mathbb{R} \\ \begin{array}{rrcl} \text{s.t.} & A^t y - cz & \geq & 0 \\ & yb^t - \alpha z & < & 0 \\ & y, z & \geq & 0 \end{array}\end{array}$$

From the definition of α we know that the first system is infeasible; hence the second must be feasible.

Proof of Strong Duality

$$\begin{array}{llcl} \exists y \in \mathbb{R}^m; z \in \mathbb{R} & & & \\ \text{s.t.} & A^t y - cz & \geq & 0 \\ & yb^t - \alpha z & < & 0 \\ & y, z & \geq & 0 \end{array}$$

If the solution y, z has $z = 0$ we have that

$$\begin{array}{llcl} \exists y \in \mathbb{R}^m & & & \\ \text{s.t.} & A^t y & \geq & 0 \\ & yb^t & < & 0 \\ & y & \geq & 0 \end{array}$$

is feasible. By Farkas lemma this gives that LP P is infeasible. Contradiction to the assumption of the lemma.

Proof of Strong Duality

Hence, there exists a solution y, z with $z > 0$.

We can rescale this solution (scaling both y and z) s.t. $z = 1$.

Then y is feasible for the dual but $b^t y < \alpha$. This means that $w < \alpha$.

Fundamental Questions

Definition 12 (Linear Programming Problem (LP))

Let $A \in \mathbb{Q}^{m\times n}$, $b \in \mathbb{Q}^m$, $c \in \mathbb{Q}^n$, $\alpha \in \mathbb{Q}$. Does there exist $x \in \mathbb{Q}^n$ s.t. $Ax = b$, $x \geq 0$, $c^t x \geq \alpha$?

Questions:

- Is LP in NP?
- Is LP in co-NP? yes!
- Is LP in P?

Proof:

- Given a primal maximization problem P and a parameter α. Suppose that $\alpha > \text{opt}(P)$.
- We can prove this by providing an optimal basis for the dual.
- A verifier can check that the associated dual solution fulfills all dual constraint and that it has dual cost $< \alpha$.

Complementary Slackness

Lemma 13

Assume a linear program $P = \max\{c^t x \mid Ax \le b; x \ge 0\}$ *has solution* x^* *and its dual* $D = \min\{b^t y \mid A^t y \ge c; y \ge 0\}$ *has solution* y^*.

1. *If* $x_j^* > 0$ *then the* j*-th constraint in* D *is tight.*
2. *If the* j*-th constraint in* D *is not tight than* $x_j^* = 0$.
3. *If* $y_i^* > 0$ *then the* i*-th constraint in* P *is tight.*
4. *If the* i*-th constraint in* P *is not tight than* $y_i^* = 0$.

If we say that a variable x_j^* (y_i^*) has slack if $x_j^* > 0$ ($y_i^* > 0$), (i.e., the corresponding variable restriction is not tight) and a contraint has slack if it is not tight, then the above says that for a primal-dual solution pair it is not possible that a constraint **and** its corresponding (dual) variable has slack.

Proof: Complementary Slackness

Analogous to the proof of weak duality we obtain

$$c^t x^* \leq y^{*t} A x^* \leq b^t y^*$$

Because of strong duality we then get

$$c^t x^* = y^{*t} A x^* = b^t y^*$$

This gives e.g.

$$\sum_j (y^t A - c^t)_j x_j^* = 0$$

From the constraint of the dual it follows that $y^t A \geq c^t$. Hence the left hand side is a sum over the product of non-negative numbers. Hence, if e.g. $(y^t A - c^t)_j > 0$ (the j-th constraint in the dual is not tight) then $x_j = 0$ (2.). The result for (1./3./4.) follows similarly.

Interpretation of Dual Variables

- Brewer: find mix of ale and beer that maximizes profits

$$\begin{array}{rrcrl} \text{max} & 13a & + & 23b & \\ \text{s.t.} & 5a & + & 15b & \le 480 \\ & 4a & + & 4b & \le 160 \\ & 35a & + & 20b & \le 1190 \\ & & & a, b & \ge 0 \end{array}$$

- Entrepeneur: buy resources from brewer at minimum cost
  C, H, M: unit price for corn, hops and malt.

$$\begin{array}{rrcrcrl} \text{min} & 480C & + & 160H & + & 1190M & \\ \text{s.t.} & 5C & + & 4H & + & 35M & \ge 13 \\ & 15C & + & 4H & + & 20M & \ge 23 \\ & & & & & C, H, M & \ge 0 \end{array}$$

  Note that brewer won't sell (at least not all) if e.g.
  $5C + 4H + 35M < 13$ as then brewing ale would be advantageous.

Interpretation of Dual Variables

Marginal Price:

- How much money is the brewer willing to pay for additional amount of Corn, Hops, or Malt?
- We are interested in the marginal price, i.e., what happens if we increase the amount of Corn, Hops, and Malt by ε_C, ε_H, and ε_M, respectively.

The profit increases to $\max\{c^t x \mid Ax \leq b + \varepsilon; x \geq 0\}$. Because of strong duality this is equal to

$$\begin{array}{lrcl} \min & (b^t + \epsilon^t)y & & \\ \text{s.t.} & A^t y & \geq & c \\ & y & \geq & 0 \end{array}$$

Interpretation of Dual Variables

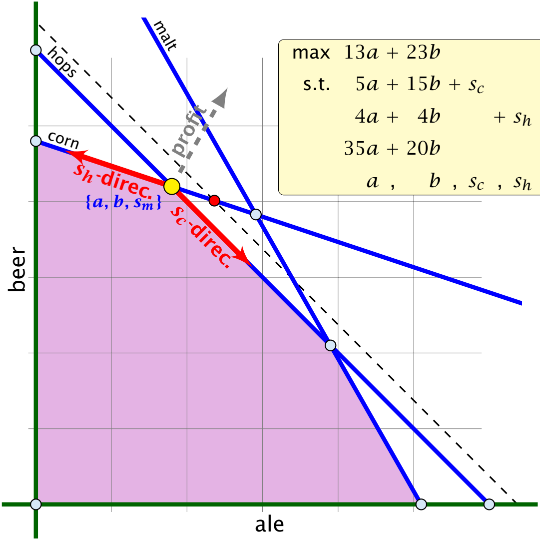
If ϵ is "small" enough then the optimum dual solution y^* might not change. Therefore the profit increases by $\sum_i \varepsilon_i y_i^*$.

Therefore we can interpret the dual variables as marginal prices.

Note that with this interpretation, complementary slackness becomes obvious.

- If the brewer has slack of some resource (e.g. corn) then he is not willing to pay anything for it (corresponding dual variable is zero).
- If the dual variable for some resource is non-zero, then an increase of this resource increases the profit of the brewer. Hence, it makes no sense to have left-overs of this resource. Therefore its slack must be zero.

Example



$$\begin{array}{lrcl}
\max & 13a + 23b & & \\
\text{s.t.} & 5a + 15b + s_c & = & 480 \\
 & 4a + 4b + s_h & = & 160 \\
 & 35a + 20b + s_m & = & 1190 \\
 & a, b, s_c, s_h, s_m & \geq & 0
\end{array}$$

The change in profit when increasing hops by one unit is

$$-\tilde{c}_h = -c_h + c_B^t A_B^{-1} A_{*h} = \underbrace{c_B^t A_B^{-1}}_{y^*} e_h .$$

Of course, the previous argument about the increase in the primal objective only holds for the non-degenerate case.

If the optimum basis is degenerate then increasing the supply of one resource may not allow the objective value to increase.

Flows

Definition 14

An (s,t)-flow in a (complete) directed graph $G = (V, V \times V, c)$ is a function $f : V \times V \mapsto \mathbb{R}_0^+$ that satisfies

1. For each edge (x,y)
$$0 \leq f_{xy} \leq c_{xy} \ .$$
(capacity constraints)

2. For each $v \in V \setminus \{s,t\}$
$$\sum_x f_{vx} = \sum_x f_{xv} \ .$$
(flow conservation constraints)

Flows

Definition 15

The value of an (s,t)-flow f is defined as

$$\operatorname{val}(f) = \sum_x f_{sx} - \sum_x f_{xs} \quad .$$

Maximum Flow Problem:

Find an (s,t)-flow with maximum value.

LP-Formulation of Maxflow

$$
\begin{array}{llrcll}
\max & & \sum_z f_{sz} - \sum_z f_{zs} & & & \\
\text{s.t.} & \forall (z,w) \in V \times V & f_{zw} & \le & c_{zw} & \ell_{zw} \\
& \forall w \neq s,t & \sum_z f_{zw} - \sum_z f_{wz} & = & 0 & p_w \\
& & f_{zw} & \ge & 0 &
\end{array}
$$

$$
\begin{array}{lllcr}
\min & & \sum_{(xy)} c_{xy} \ell_{xy} & & \\
\text{s.t.} & f_{xy}\ (x, y \neq s,t): & 1\ell_{xy} - 1p_x + 1p_y & \ge & 0 \\
& f_{sy}\ (y \neq s,t): & 1\ell_{sy} + 1p_y & \ge & 1 \\
& f_{xs}\ (x \neq s,t): & 1\ell_{xs} - 1p_x & \ge & -1 \\
& f_{ty}\ (y \neq s,t): & 1\ell_{ty} + 1p_y & \ge & 0 \\
& f_{xt}\ (x \neq s,t): & 1\ell_{xt} - 1p_x & \ge & 0 \\
& f_{st}: & 1\ell_{st} & \ge & 1 \\
& f_{ts}: & 1\ell_{ts} & \ge & -1 \\
& & \ell_{xy} & \ge & 0
\end{array}
$$

LP-Formulation of Maxflow

$$
\begin{array}{llrcl}
\min & & \sum_{(xy)} c_{xy}\ell_{xy} & & \\
\text{s.t.} & f_{xy}\ (x,y \neq s,t): & 1\ell_{xy} - 1p_x + 1p_y & \geq & 0 \\
 & f_{sy}\ (y \neq s,t): & 1\ell_{sy} - 1 + 1p_y & \geq & 0 \\
 & f_{xs}\ (x \neq s,t): & 1\ell_{xs} - 1p_x + 1 & \geq & 0 \\
 & f_{ty}\ (y \neq s,t): & 1\ell_{ty} - 0 + 1p_y & \geq & 0 \\
 & f_{xt}\ (x \neq s,t): & 1\ell_{xt} - 1p_x + 0 & \geq & 0 \\
 & f_{st}: & 1\ell_{st} - 1 + 0 & \geq & 0 \\
 & f_{ts}: & 1\ell_{ts} - 0 + 1 & \geq & 0 \\
 & & \ell_{xy} & \geq & 0
\end{array}
$$

LP-Formulation of Maxflow

$$
\begin{array}{llrcl}
\min & & \sum_{(xy)} c_{xy}\ell_{xy} & & \\
\text{s.t.} & f_{xy}\ (x,y \neq s,t): & 1\ell_{xy} - 1p_x + 1p_y & \geq & 0 \\
 & f_{sy}\ (y \neq s,t): & 1\ell_{sy} - p_s + 1p_y & \geq & 0 \\
 & f_{xs}\ (x \neq s,t): & 1\ell_{xs} - 1p_x + p_s & \geq & 0 \\
 & f_{ty}\ (y \neq s,t): & 1\ell_{ty} - p_t + 1p_y & \geq & 0 \\
 & f_{xt}\ (x \neq s,t): & 1\ell_{xt} - 1p_x + p_t & \geq & 0 \\
 & f_{st}: & 1\ell_{st} - p_s + p_t & \geq & 0 \\
 & f_{ts}: & 1\ell_{ts} - p_t + p_s & \geq & 0 \\
 & & \ell_{xy} & \geq & 0
\end{array}
$$

with $p_t = 0$ and $p_s = 1$.

LP-Formulation of Maxflow

$$\begin{array}{llrcl} \min & & \sum_{(xy)} c_{xy}\ell_{xy} & & \\ \text{s.t.} & f_{xy}: & 1\ell_{xy} - 1p_x + 1p_y & \geq & 0 \\ & & \ell_{xy} & \geq & 0 \\ & & p_s & = & 1 \\ & & p_t & = & 0 \end{array}$$

We can interpret the ℓ_{xy} value as assigning a length to every edge.

The value p_x for a variable, then can be seen as the distance of x to t (where the distance from s to t is required to be 1 since $p_s = 1$).

The constraint $p_x \leq \ell_{xy} + p_y$ then simply follows from triangle inequality ($d(x,t) \leq d(x,y) + d(y,t) \Rightarrow d(x,t) \leq \ell_{xy} + d(y,t)$).

One can show that there is an optimum LP-solution for the dual problem that gives an integral assignment of variables.

This means $p_x = 1$ or $p_x = 0$ for our case. This gives rise to a cut in the graph with vertices having value 1 on one side and the other vertices on the other side. The objective function then evaluates the capacity of this cut.

This shows that the Maxflow/Mincut theorem follows from linear programming duality.