Technique 2: Rounding the Dual Solution.

Relaxation for Set Cover

Primal:

$$\begin{array}{lll} \min & \sum_{i\in I} w_i x_i & \\ \text{s.t. } \forall u & \sum_{i:u\in S_i} x_i & \geq 1 \\ & x_i & \geq 0 \end{array}$$

Dual:

$$\begin{array}{lll} \max & \sum_{u\in U} y_u & \\ \text{s.t. } \forall i & \sum_{u:u\in S_i} y_u & \leq w_i \\ & y_u & \geq 0 \end{array}$$

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Rounding Algorithm:

Let I denote the index set of sets for which the dual constraint is tight. This means for all $i \in I$

$$\sum_{u:u\in S_i} y_u = w_i$$

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Lemma 3

The resulting index set is an f-approximation.

Proof:

Every $u \in U$ is covered.

- Suppose there is a u that is not covered.
- This means $\sum_{u: u \in S_j} y_u < w_j$ for all sets S_j that contain u.
- But then y_u could be increased in the dual solution without violating any constraint. This is a contradiction to the fact that the dual solution is optimal.

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$$\begin{aligned}\sum_{i\in I} w_i &= \sum_{i\in I}\sum_{u:u\in S_i} y_u \\ &= \sum_u |\{i\in I : u\in S_i\}|\cdot y_u \\ &\le \sum_u f_u y_u\end{aligned}$$

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&\le f \mathrm{cost}(x^*) \\
&\le f \cdot \mathrm{OPT}
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Let I denote the solution obtained by the first rounding algorithm and I' be the solution returned by the second algorithm. Then

$$I \subseteq I' \ .$$

This means I' is never better than I.

Suppose that we take S_j in the first algorithm. I.e., $j \in I$.

This means $x_j \geq \frac{1}{f}$.

Because of Complementary Slackness Conditions the corresponding constraint in the dual must be tight.

Hence, the second algorithm will also choose S_j.

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