1 Planar graph coloring

1.1 Problem

In this section, we consider planar graph coloring. This problem is relevant for applications such as:

- Can a map be colored with only 4 colors, such that neighboring regions have different colors?
- How does one have to assign phone/radio frequencies to antennas to use as few frequencies as possible?
- Sudoku.

A coloring of an undirected graph $G = (V, E)$ assigns to each vertex v (edge e, face or region f) from V (E, F) a natural number $c(v) \in \mathbb{N}$ ($c(e)$, $c(f)$), called *color*. A coloring is *feasible* if two neighboring vertices (edges, faces) are assigned different colors. One distinguishes between vertex coloring, edge coloring and face coloring. A coloring using k different colors is called k-coloring. $\gamma(G) = \min\{k : G \text{ is } k\text{-colorable}\}$ is called *chromatic number*.

In general, the decision problem regarding the k-colorability of a given graph G is NP-hard.

Definition 1 (planar graph) *A graph G is planar, if the vertices can be embedded into the plane $\mathbb{R}^2$ such that edges can only cross each other at a vertex.*

We consider vertex coloring of a planar graph G. The following result will be helpful:

Theorem 2 (Euler's formula) *Given a connected planar graph G, let n, e, f be the number of vertices, edges, and faces, respectively. Then*

$$n - e + f = 2.$$

Proof: Proof by induction over n.
Let $n = 1$. Then G contains only loops. If $e = 0$, then there is exactly one face. If $e > 0$, then each loop divides a face into two faces. Thus, the induction hypothesis is true for $n = 1$ and $e \geq 0$.

Let $n > 1$. If G is connected, there exists an edge which is not a loop. Then this edge can be "collapsed" resulting in a graph G' containing n' vertices, e' edges and f' faces. By collapsing the edge, the number of faces does not change. However, the number of vertices and the number edges each decrease by 1. Using the hypothesis yields

$$n - e + f = n' + 1 - (e' + 1) + f = 2.$$

□

Remark 3 *Euler's formula only works for connected graphs. However, there is a general formula for planar graphs with k connected components:*

$$n - e + f = k + 1.$$

Theorem 4 (Five color theorem (Heawood)) *Every planar graph is 5-colorable.*

Proof: The case $|V| \leq 5$ is trivial.
Now let $G = (V, E)$ be a minimal non-5-colorable planar graph. Since G is planar, it can be triangulated, and we have $e \leq \frac{3f}{2}$. Substituting this to Euler's formula we get $e \leq 3n - 6$. The average degree of a vertex in G can be computed as

$$a = \frac{2e}{n} \leq \frac{2(3n-6)}{n} = \frac{6n-12}{6} < 6.$$

Therefore there is a vertex $w \in V$ with $deg(w) \leq 5$. And because a vertex with degree 4 can always be colored with a color that is not used by its neighbors (in contradiction to the minimality), this vertex must have degree 5. Let $v \in V$ be such a vertex.

Because of the minimality of G, $G \backslash \{v\}$ is 5-colorable. So, let $f : V \backslash \{v\} \to \{1, \ldots, 5\}$ be a 5-coloring of $G \backslash \{v\}$. Because G is not 5-colorable, every vertex in the neighborhood $N(v)$ of v is assigned a different color. We color the neighbors $v_1, \ldots, v_5$ clockwise with the colors $1, \ldots, 5$.

Let $G_{1,3}$ be the subgraph of G containing all vertices which are colored with the colors 1 and 3. Changing the colors of the two components in this subgraph obtains another 5-coloring of $G \setminus \{v\}$. If v_1 and v_3 lie in different components, we can change the coloring in that containing v_1 such that v_1 is now assigned the color 3 and G is 5-colorable. Thus, v_1 and v_3 must lie in the same component.

Let $P_{1,3}$ be the path in $G_{1,3}$, starting at v_1 and ending at v_3. Consider the cycle defined by the vertices in $P_{1,3}$ and v. This cycle divides the vertices v_2 and v_4. Thus, the path $P_{2,4}$ must cross $P_{1,3}$. As G is planar, the crossing must be at a vertex $v \in V$. But $P_{1,3}$ and $P_{2,4}$ do not share any color, thus the crossing cannot be at a vertex, which is a contradiction. □

Theorem 5 (Four color theorem) *Every planar graph is 4-colorable.*

Remark: Not until 1977, Ken Appel and Wolfgang Haken were able to find a proof. This proof reduced the number of problematic cases from infinitely many to 1936 (and in a later version even to 1476), which were checked individually by a computer. In 1996, Neil Robertson, Daniel Sanders, Paul Seymour, and Robin Thomas found a modified proof reducing the cases to 633. But these had to be checked by a computer, as well.

1.2 Algorithm

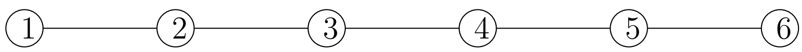
Consider the greedy approach given in figure 1:

func greedy coloring 1$((V, E))$

$\forall v \in V$ do
  color v with the next free color
od

Abbildung 1: Greedy coloring 1

It is obvious that this algorithm finds a feasible coloring of G, in time $O(|V| + |E|)$. But this coloring can become arbitrarily bad, see the following example: Although 2 colors are sufficient the algorithm uses 6 colors.



The algorithm in figure 1 can be improved by a simple modification, see the algorithm in figure 2.

func greedy coloring 2$((V, E))$

$\forall v \in V$ do
  $c[v] = \min\{k \in \mathbb{N} \mid k \neq c[w] \quad \forall e = \{v, w\}\}$
od

Abbildung 2: Greedy coloring 2

In general, the algorithm in figure 2 uses less colors than the algorithm in figure 1, requiring the same time $O(|V| + |E|)$. In the example above, it uses only 2 colors. But still, the coloring can become arbitrarily bad.

Lemma 6 *Every graph G of maximal degree d can be colored with $d + 1$ colors.*

Lemma 7 *For every graph $G = (V, E)$ there exists an ordering σ of the vertices $v \in V$ such that the greedy algorithm in figure 2 assigns an optimal coloring.*

Unfortunately, we cannot determine an optimal ordering in advance. In some cases, it is sufficient to determine a "good" coloring of G, and thus a "good" ordering of the vertices. If we consider a vertex $v \in V$, which we would like to color, then the colors $c[v_i]$ of its neighbors in $N(v)$ are "forbidden". Thus, for a vertex with a big degree, many colors are "forbidden". Vertices with lower degree allow for a more flexible choice of colors. Therefore it is intuitive to color the other vertices (i.e. vertices with high degree) with higher priority.

Because of this, we can come up with algorithm 3. This algorithm generally computes a "good" coloring of a graph G in time $O(|V|\log|V|+|E|)$:

func greedy coloring 3$((V,E))$

Sort the vertices $v \in V$ according to non-increasing degree $deg(v) \to \sigma[.]$

for $i = 1$ to $|\sigma|$ do
$c[\sigma[i]] = \min\{k \in \mathbb{N} \mid k \neq c[w] \quad \forall e = \{\sigma[i], w\}\}$
od

Abbildung 3: Greedy coloring 3

Literatur

[1] K. Appel and W. Haken, *Every Planar Map is Four Colorable, Part I: Discharging.* Illinois J. Math., 21, 429-490, 1977.

[2] L. Euler, *Demonstratio Nonullarum Insignium Proprietatum Quibus Solida Hedris Planis Inculsa Sunt Praedita.* Novi Comm. Acad. Sci. Imp. Petrol, 4, 140-160, 1758.

[3] P.J. Heawood, *Map-Color Theorem.* Q. J. Math., 24, 332-339, 1890.