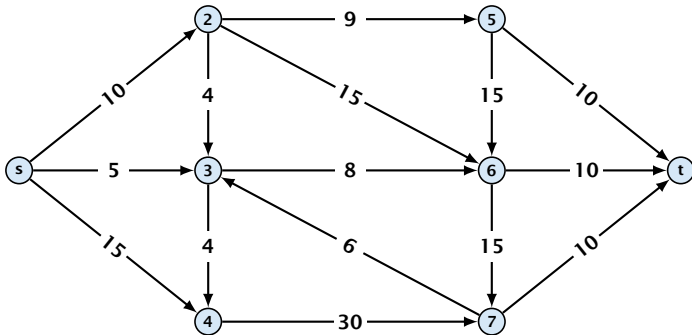


# 11 Introduction

## Flow Network

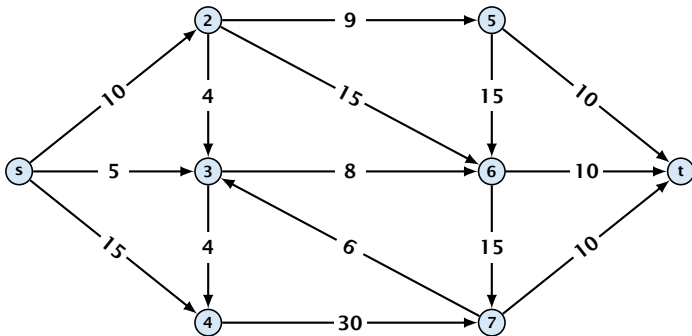
- ▶ directed graph  $G = (V, E)$ ; edge capacities  $c(e)$
- ▶ two special nodes: source  $s$ ; target  $t$ ;
- ▶ no edges entering  $s$  or leaving  $t$ ;
- ▶ at least for now: no parallel edges;



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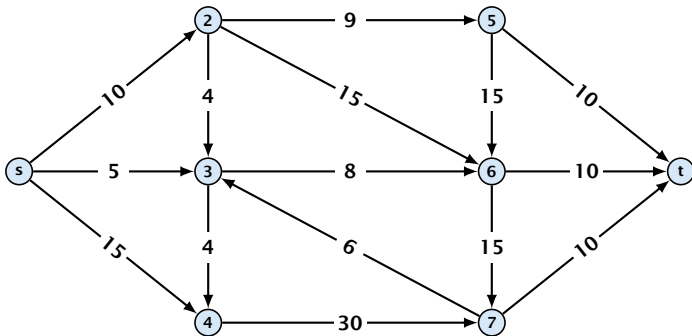
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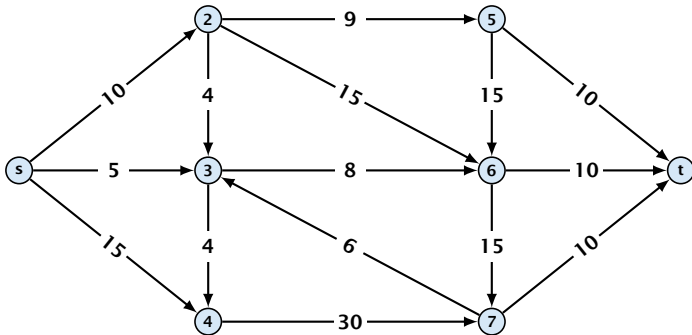
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The **capacity** of a cut  $A$  is defined as

$$\text{cap}(A, V \setminus A) := \sum_{e \in \text{out}(A)} c(e) ,$$

where  $\text{out}(A)$  denotes the set of edges of the form  $A \times V \setminus A$  (i.e. edges leaving  $A$ ).

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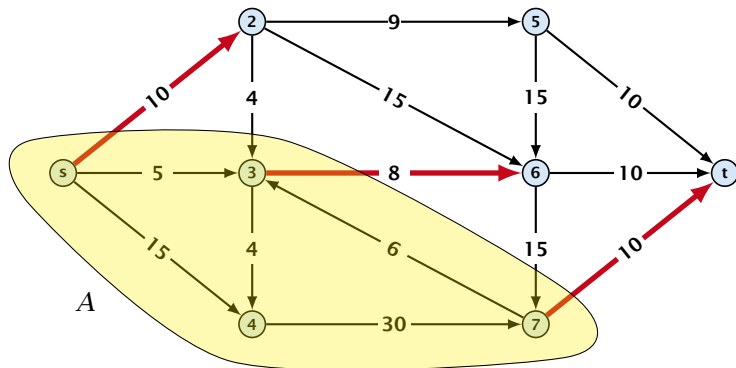
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**Minimum Cut Problem:** Find an  $(s, t)$ -cut with minimum capacity.

# Cuts

## Example 3



The capacity of the cut is  $\text{cap}(A, V \setminus A) = 28$ .



## Definition 4

An  $(s, t)$ -flow is a function  $f : E \mapsto \mathbb{R}^+$  that satisfies

1. For each edge  $e$

$$0 \leq f(e) \leq c(e) .$$

(capacity constraints)

2. For each  $v \in V \setminus \{s, t\}$

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## Definition 5

The **value of an  $(s, t)$ -flow  $f$**  is defined as

$$\text{val}(f) = \sum_{e \in \text{out}(s)} f(e) .$$

**Maximum Flow Problem:** Find an  $(s, t)$ -flow with maximum value.

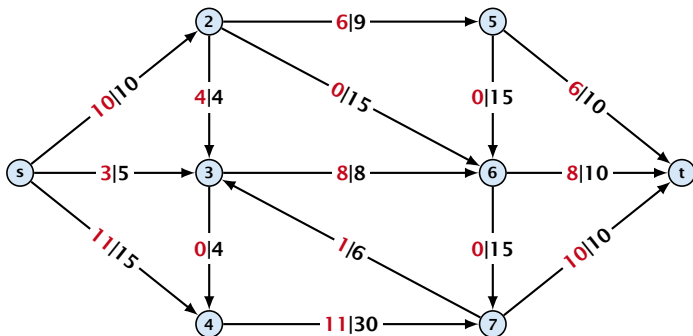
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## Example 6



The value of the flow is  $\text{val}(f) = 24$ .

## Lemma 7 (Flow value lemma)

Let  $f$  a flow, and let  $A \subseteq V$  be an  $(s, t)$ -cut. Then the *net-flow* across the cut is equal to the amount of flow leaving  $s$ , i.e.,

$$\text{val}(f) = \sum_{e \in \text{out}(A)} f(e) - \sum_{e \in \text{into}(A)} f(e) .$$

**Proof.**

$\text{val}(f)$

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$$\begin{aligned}\text{val}(f) &= \sum_{e \in \text{out}(s)} f(e) \\ &= \sum_{e \in \text{out}(s)} f(e) + \sum_{v \in A \setminus \{s\}} \left( \sum_{e \in \text{out}(v)} f(e) - \sum_{e \in \text{in}(v)} f(e) \right)\end{aligned}$$

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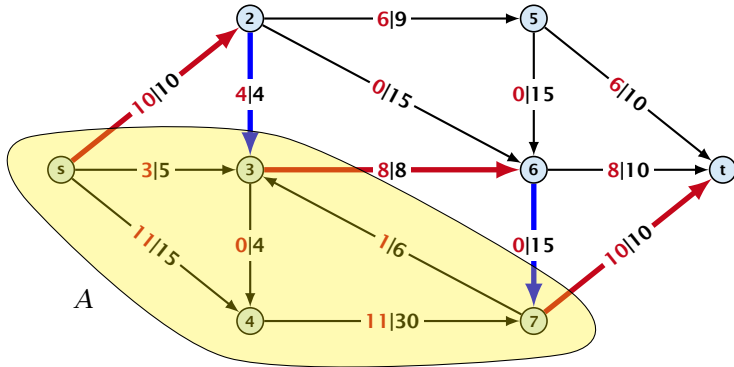
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The last equality holds since every edge with both end-points in  $A$  contributes negatively as well as positively to the sum in Line 2. The only edges whose contribution doesn't cancel out are edges leaving or entering  $A$ . □

## Example 8



## Corollary 9

*Let  $f$  be an  $(s, t)$ -flow and let  $A$  be an  $(s, t)$ -cut, such that*

$$\text{val}(f) = \text{cap}(A, V \setminus A).$$

*Then  $f$  is a maximum flow.*

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Suppose that there is a flow  $f'$  with larger value. Then

$$\text{cap}(A, V \setminus A) < \text{val}(f')$$



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Suppose that there is a flow  $f'$  with larger value. Then

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