

6.2 Master Theorem

Note that the cases do not cover all possibilities.

Lemma 1

Let $a \geq 1$, $b \geq 1$ and $\epsilon > 0$ denote constants. Consider the recurrence

$$T(n) = aT\left(\frac{n}{b}\right) + f(n).$$

Case 1.

If $f(n) = \mathcal{O}(n^{\log_b(a) - \epsilon})$ then $T(n) = \Theta(n^{\log_b a})$.

Case 2.

If $f(n) = \Theta(n^{\log_b(a)} \log^k n)$ then $T(n) = \Theta(n^{\log_b a} \log^{k+1} n)$, $k \geq 0$.

Case 3.

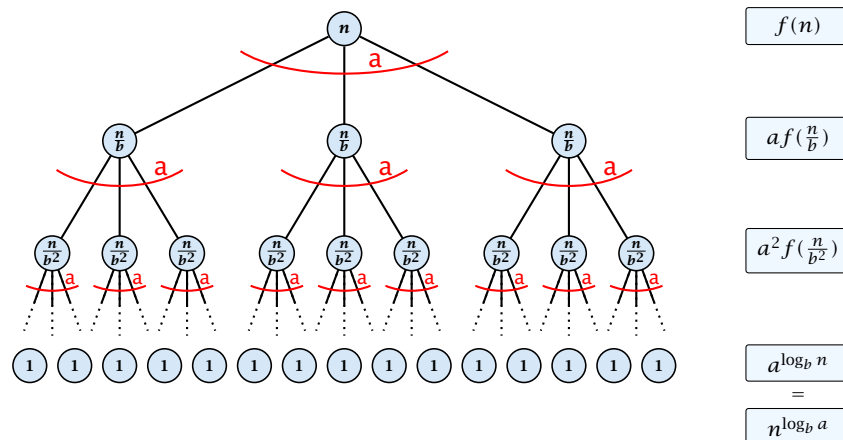
If $f(n) = \Omega(n^{\log_b(a) + \epsilon})$ and for sufficiently large n $af\left(\frac{n}{b}\right) \leq cf(n)$ for some constant $c < 1$ then $T(n) = \Theta(f(n))$.

6.2 Master Theorem

We prove the Master Theorem for the case that n is of the form b^ℓ , and we assume that the non-recursive case occurs for problem size 1 and incurs cost 1.

The Recursion Tree

The running time of a recursive algorithm can be visualized by a recursion tree:



6.2 Master Theorem

This gives

$$T(n) = n^{\log_b a} + \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right).$$

Case 1. Now suppose that $f(n) \leq cn^{\log_b a - \epsilon}$.

$$T(n) - n^{\log_b a} = \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) \leq c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a - \epsilon}$$

$$\begin{aligned} \boxed{b^{-i(\log_b a - \epsilon)} = b^{\epsilon i} (b^{\log_b a})^{-i} = b^{\epsilon i} a^{-i}} &= cn^{\log_b a - \epsilon} \sum_{i=0}^{\log_b n - 1} (b^\epsilon)^i \\ \boxed{\sum_{i=0}^k q^i = \frac{q^{k+1} - 1}{q - 1}} &= cn^{\log_b a - \epsilon} (b^{\epsilon \log_b n} - 1) / (b^\epsilon - 1) \\ &= cn^{\log_b a - \epsilon} (n^\epsilon - 1) / (b^\epsilon - 1) \\ &= \frac{c}{b^\epsilon - 1} n^{\log_b a} (n^\epsilon - 1) / (n^\epsilon) \end{aligned}$$

Hence,

$$T(n) \leq \left(\frac{c}{b^\epsilon - 1} + 1\right) n^{\log_b a} \Rightarrow T(n) = \mathcal{O}(n^{\log_b a}).$$

Case 2. Now suppose that $f(n) \leq cn^{\log_b a}$.

$$\begin{aligned} T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) \\ &\leq c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a} \\ &= cn^{\log_b a} \sum_{i=0}^{\log_b n - 1} 1 \\ &= cn^{\log_b a} \log_b n \end{aligned}$$

Hence,

$$T(n) = \mathcal{O}(n^{\log_b a} \log_b n) \Rightarrow T(n) = \mathcal{O}(n^{\log_b a} \log n).$$

Case 2. Now suppose that $f(n) \geq cn^{\log_b a}$.

$$\begin{aligned} T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) \\ &\geq c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a} \\ &= cn^{\log_b a} \sum_{i=0}^{\log_b n - 1} 1 \\ &= cn^{\log_b a} \log_b n \end{aligned}$$

Hence,

$$T(n) = \Omega(n^{\log_b a} \log_b n) \Rightarrow T(n) = \Omega(n^{\log_b a} \log n).$$

Case 2. Now suppose that $f(n) \leq cn^{\log_b a} (\log_b(n))^k$.

$$\begin{aligned} T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) \\ &\leq c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a} \cdot \left(\log_b \left(\frac{n}{b^i}\right)\right)^k \\ \boxed{n = b^\ell \Rightarrow \ell = \log_b n} &= cn^{\log_b a} \sum_{i=0}^{\ell - 1} \left(\log_b \left(\frac{b^\ell}{b^i}\right)\right)^k \\ &= cn^{\log_b a} \sum_{i=0}^{\ell - 1} (\ell - i)^k \\ &= cn^{\log_b a} \sum_{i=1}^{\ell} i^k \approx \frac{1}{k} \ell^{k+1} \\ &\approx \frac{c}{k} n^{\log_b a} \ell^{k+1} \Rightarrow T(n) = \mathcal{O}(n^{\log_b a} \log^{k+1} n). \end{aligned}$$

Case 3. Now suppose that $f(n) \geq dn^{\log_b a + \epsilon}$, and that for sufficiently large n : $af(n/b) \leq cf(n)$, for $c < 1$.

From this we get $a^i f(n/b^i) \leq c^i f(n)$, where we assume that $n/b^{i-1} \geq n_0$ is still sufficiently large.

$$T(n) - n^{\log_b a} = \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) \leq \sum_{i=0}^{\log_b n - 1} c^i f(n) + \mathcal{O}(n^{\log_b a})$$

$$q < 1 : \sum_{i=0}^n q^i = \frac{1 - q^{n+1}}{1 - q} \leq \frac{1}{1 - q} \leq \frac{1}{1 - c} \Rightarrow T(n) \leq \frac{1}{1 - c} f(n) + \mathcal{O}(n^{\log_b a})$$

Hence,

$$T(n) \leq \mathcal{O}(f(n)) \Rightarrow T(n) = \Theta(f(n)).$$

Where did we use $f(n) \geq \Omega(n^{\log_b a + \epsilon})$?

Example: Multiplying Two Integers

Suppose we want to multiply two n -bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers A and B :

$$\begin{array}{r} 110110101 \quad A \\ + 100010011 \quad B \\ \hline 10111001000 \end{array}$$

This gives that two n -bit integers can be added in time $\mathcal{O}(n)$.

Example: Multiplying Two Integers

Suppose that we want to multiply an n -bit integer A and an m -bit integer B ($m \leq n$).

$$\begin{array}{r} 10001 \times 1011 \\ \hline 10001 \\ 100010 \\ 0000000 \\ 10001000 \\ \hline 10111011 \end{array}$$

- This is also known as the "school method" for multiplying integers.
- Note that the intermediate numbers that are generated can have at most $m + n \leq 2n$ bits.

Time requirement:

- ▶ Computing intermediate results: $\mathcal{O}(nm)$.
- ▶ Adding m numbers of length $\leq 2n$: $\mathcal{O}((m + n)m) = \mathcal{O}(nm)$.

Example: Multiplying Two Integers

A recursive approach:

Suppose that integers A and B are of length $n = 2^k$, for some k .

$$\boxed{B_1} \boxed{B_0} \times \boxed{A_1} \boxed{A_0}$$

Then it holds that

$$A = A_1 \cdot 2^{\frac{n}{2}} + A_0 \quad \text{and} \quad B = B_1 \cdot 2^{\frac{n}{2}} + B_0$$

Hence,

$$A \cdot B = A_1 B_1 \cdot 2^n + (A_1 B_0 + A_0 B_1) \cdot 2^{\frac{n}{2}} + A_0 \cdot B_0$$

Example: Multiplying Two Integers

Algorithm 3 mult(A, B)

1: if $ A = B = 1$ then	$\mathcal{O}(1)$
2: return $a_0 \cdot b_0$	$\mathcal{O}(1)$
3: split A into A_0 and A_1	$\mathcal{O}(n)$
4: split B into B_0 and B_1	$\mathcal{O}(n)$
5: $Z_2 \leftarrow \text{mult}(A_1, B_1)$	$T(\frac{n}{2})$
6: $Z_1 \leftarrow \text{mult}(A_1, B_0) + \text{mult}(A_0, B_1)$	$2T(\frac{n}{2}) + \mathcal{O}(n)$
7: $Z_0 \leftarrow \text{mult}(A_0, B_0)$	$T(\frac{n}{2})$
8: return $Z_2 \cdot 2^n + Z_1 \cdot 2^{\frac{n}{2}} + Z_0$	$\mathcal{O}(n)$

We get the following recurrence:

$$T(n) = 4T\left(\frac{n}{2}\right) + \mathcal{O}(n) .$$

Example: Multiplying Two Integers

Master Theorem: Recurrence: $T[n] = aT(\frac{n}{b}) + f(n)$.

- ▶ Case 1: $f(n) = \mathcal{O}(n^{\log_b a - \epsilon})$ $T(n) = \Theta(n^{\log_b a})$
- ▶ Case 2: $f(n) = \Theta(n^{\log_b a} \log^k n)$ $T(n) = \Theta(n^{\log_b a} \log^{k+1} n)$
- ▶ Case 3: $f(n) = \Omega(n^{\log_b a + \epsilon})$ $T(n) = \Theta(f(n))$

In our case $a = 4$, $b = 2$, and $f(n) = \Theta(n)$. Hence, we are in Case 1, since $n = \mathcal{O}(n^{2-\epsilon}) = \mathcal{O}(n^{\log_b a - \epsilon})$.

We get a running time of $\mathcal{O}(n^2)$ for our algorithm.

⇒ Not better than the “school method”.

Example: Multiplying Two Integers

We can use the following identity to compute Z_1 :

$$\begin{aligned} Z_1 &= A_1 B_0 + A_0 B_1 && = Z_2 && = Z_0 \\ &= (A_0 + A_1) \cdot (B_0 + B_1) - \underbrace{A_1 B_1}_{Z_2} - \underbrace{A_0 B_0}_{Z_0} \end{aligned}$$

Hence,

Algorithm 4 mult(A, B)

1: if $ A = B = 1$ then	$\mathcal{O}(1)$
2: return $a_0 \cdot b_0$	$\mathcal{O}(1)$
3: split A into A_0 and A_1	$\mathcal{O}(n)$
4: split B into B_0 and B_1	$\mathcal{O}(n)$
5: $Z_2 \leftarrow \text{mult}(A_1, B_1)$	$T(\frac{n}{2})$
6: $Z_0 \leftarrow \text{mult}(A_0, B_0)$	$T(\frac{n}{2})$
7: $Z_1 \leftarrow \text{mult}(A_0 + A_1, B_0 + B_1) - Z_2 - Z_0$	$T(\frac{n}{2}) + \mathcal{O}(n)$
8: return $Z_2 \cdot 2^n + Z_1 \cdot 2^{\frac{n}{2}} + Z_0$	$\mathcal{O}(n)$

A more precise (correct) analysis would say that computing Z_1 needs time $T(\frac{n}{2} + 1) + \mathcal{O}(n)$.

Example: Multiplying Two Integers

We get the following recurrence:

$$T(n) = 3T\left(\frac{n}{2}\right) + \mathcal{O}(n) .$$

Master Theorem: Recurrence: $T[n] = aT(\frac{n}{b}) + f(n)$.

- ▶ Case 1: $f(n) = \mathcal{O}(n^{\log_b a - \epsilon})$ $T(n) = \Theta(n^{\log_b a})$
- ▶ Case 2: $f(n) = \Theta(n^{\log_b a} \log^k n)$ $T(n) = \Theta(n^{\log_b a} \log^{k+1} n)$
- ▶ Case 3: $f(n) = \Omega(n^{\log_b a + \epsilon})$ $T(n) = \Theta(f(n))$

Again we are in Case 1. We get a running time of $\Theta(n^{\log_2 3}) \approx \Theta(n^{1.59})$.

A huge improvement over the “school method”.