

Technique 5: Randomized Rounding

One round of randomized rounding:

Pick set S_j uniformly at random with probability $1 - x_j$ (for all j).

Version A: Repeat rounds until you have a cover.

Version B: Repeat for s rounds. If you have a cover STOP.
Otherwise, repeat the whole algorithm.

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Probability that $u \in U$ is not covered (after ℓ rounds):

$$\Pr[u \text{ not covered after } \ell \text{ round}] \leq \frac{1}{e^\ell} .$$

$\Pr[\exists u \in U \text{ not covered after } \ell \text{ round}]$

$$\begin{aligned} & \Pr[\exists u \in U \text{ not covered after } \ell \text{ round}] \\ &= \Pr[u_1 \text{ not covered} \vee u_2 \text{ not covered} \vee \dots \vee u_n \text{ not covered}] \end{aligned}$$

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Lemma 5

With high probability $\mathcal{O}(\log n)$ rounds suffice.

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With high probability:

For any constant α the number of rounds is at most $\mathcal{O}(\log n)$ with probability at least $1 - n^{-\alpha}$.

Proof: We have

$$\Pr[\text{\#rounds} \geq (\alpha + 1) \ln n] \leq ne^{-(\alpha+1)\ln n} = n^{-\alpha} .$$

Expected Cost

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Repeat for $s = (\alpha + 1) \ln n$ rounds. If you don't have a cover simply take for each element u the cheapest set that contains u .

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$$E[\text{cost} \mid \text{success}] \\ = \frac{1}{\Pr[\text{succ.}]} \left(E[\text{cost}] - \Pr[\text{no success}] \cdot E[\text{cost} \mid \text{no success}] \right)$$

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for $n \geq 2$ and $\alpha \geq 1$.

Randomized rounding gives an $\mathcal{O}(\log n)$ approximation. The running time is polynomial with high probability.

Theorem 6 (without proof)

There is no approximation algorithm for set cover with approximation guarantee better than $\frac{1}{2} \log n$ unless NP has quasi-polynomial time algorithms (algorithms with running time $2^{\text{poly}(\log n)}$).

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Integrality Gap

The **integrality gap** of the SetCover LP is $\Omega(\log n)$.

- ▶ $n = 2^k - 1$
- ▶ Elements are all vectors \mathbf{i} over $GF[2]$ of length k (excluding zero vector).
- ▶ Every vector \mathbf{j} defines a set as follows

$$S_j := \{\mathbf{i} \mid \mathbf{i} \cdot \mathbf{j} = 1\}$$

- ▶ each set contains 2^{k-1} vectors; each vector is contained in 2^{k-1} sets
- ▶ $x_i = \frac{1}{2^{k-1}} = \frac{2}{n+1}$ is fractional solution.

Integrality Gap

Every collection of $p < k$ sets does not cover all elements.

Hence, we get a gap of $\Omega(\log n)$.

Techniques:

- ▶ Deterministic Rounding
- ▶ Rounding of the Dual
- ▶ Primal Dual
- ▶ Greedy
- ▶ Randomized Rounding
- ▶ Local Search
- ▶ Rounding Data + Dynamic Programming