Technique 2: Rounding the Dual Solution.

Relaxation for Set Cover

Primal:

$$\min \quad \sum_{i \in I} w_i x_i$$
$$\text{s.t. } \forall u \quad \sum_{i:u \in S_i} x_i \geq 1$$
$$x_i \geq 0$$

Dual:

$$\max \quad \sum_{u \in U} y_u$$
$$\text{s.t. } \forall i \quad \sum_{u:u \in S_i} y_u \leq w_i$$
$$y_u \geq 0$$

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Rounding Algorithm:

Let I denote the index set of sets for which the dual constraint is tight. This means for all $i \in I$

$$\sum_{u:u \in S_i} y_u = w_i$$

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Lemma 3

The resulting index set is an f-approximation.

Proof:

Every $u \in U$ is covered.

- Suppose there is a u that is not covered.
- This means $\sum_{u : u \in S_i} y_u < w_i$ for all sets S_i that contain u.
- But then y_u could be increased in the dual solution without violating any constraint. This is a contradiction to the fact that the dual solution is optimal.

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Proof:

$$\sum_{i \in I} w_i = \sum_{i \in I} \sum_{u : u \in S_i} y_u$$

$$= \sum_u |\{i \in I : u \in S_i\}| \cdot y_u$$

$$\leq \sum_u f_u y_u$$

$$\leq f \sum_u y_u$$

$$\leq f \, \text{cost}(x^*)$$

$$\leq f \cdot \text{OPT}$$

Let I denote the solution obtained by the first rounding algorithm and I' be the solution returned by the second algorithm. Then

$$I \subseteq I' \ .$$

This means I' is never better than I.

- ▶ Suppose that we take S_i in the first algorithm. I.e., $i \in I$.
- ▶ This means $x_i \geq \frac{1}{f}$.
- ▶ Because of Complementary Slackness Conditions the corresponding constraint in the dual must be tight.
- ▶ Hence, the second algorithm will also choose S_i.