

7.7 Hashing

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- ▶ **S . insert(x)**: Insert an element x .
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So far we have implemented the search for a key by carefully choosing split-elements.

Then the memory location of an object x with key k is determined by successively comparing k to split-elements.

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Definitions:

- ▶ Universe U of keys, e.g., $U \subseteq \mathbb{N}_0$. U very large.
- ▶ Set $S \subseteq U$ of keys, $|S| = m \leq |U|$.
- ▶ Array $T[0, \dots, n-1]$ hash-table.
- ▶ Hash function $h : U \rightarrow [0, \dots, n-1]$.

The hash-function h should fulfill:

- ▶ Fast to evaluate.
- ▶ Small storage requirement.
- ▶ Good distribution of elements over the whole table.

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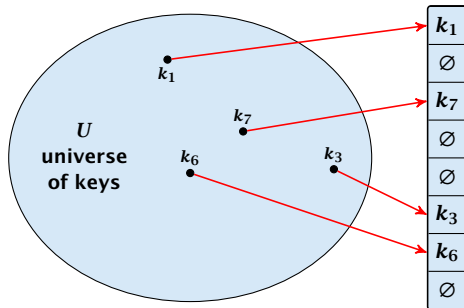
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Direct Addressing

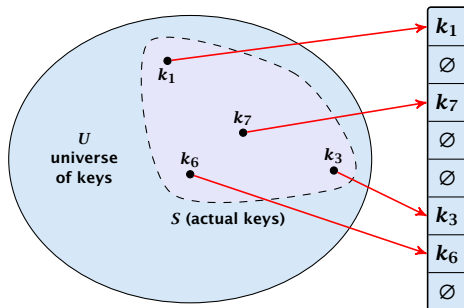
Ideally the hash function maps **all** keys to different memory locations.



This special case is known as **Direct Addressing**. It is usually very unrealistic as the universe of keys typically is quite large, and in particular larger than the available memory.

Perfect Hashing

Suppose that we **know** the set S of actual keys (no insert/no delete). Then we may want to design a **simple** hash-function that maps all these keys to different memory locations.



Such a hash function h is called a **perfect hash function** for set S .

Collisions

If we do not know the keys in advance, the best we can hope for is that the hash function distributes keys evenly across the table.

Problem: Collisions

Usually the universe U is much larger than the table-size n .

Hence, there may be two elements k_1, k_2 from the set S that map to the same memory location (i.e., $h(k_1) = h(k_2)$). This is called a **collision**.

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Typically, collisions do not appear once the size of the set S of actual keys gets close to n , but already when $|S| \geq \omega(\sqrt{n})$.

Lemma 1

The probability of having a collision when hashing m elements into a table of size n under uniform hashing is at least

$$1 - e^{-\frac{m(m-1)}{2n}} \approx 1 - e^{-\frac{m^2}{2n}}.$$

Uniform hashing:

Choose a hash function uniformly at random from all functions $f: U \rightarrow [0, \dots, n-1]$.

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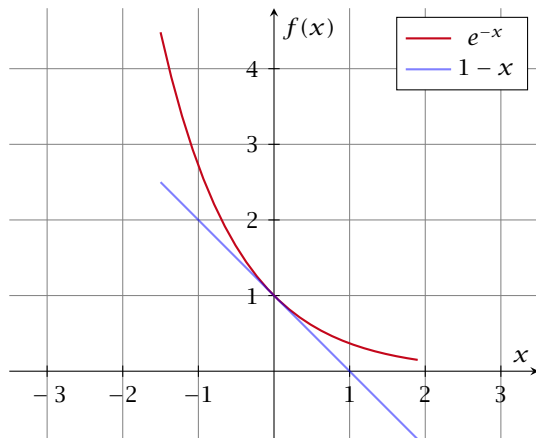
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Here the first equality follows since the ℓ -th element that is hashed has a probability of $\frac{n-\ell+1}{n}$ to not generate a collision under the condition that the previous elements did not induce collisions. □

Collisions



The inequality $1 - x \leq e^{-x}$ is derived by stopping the Taylor-expansion of e^{-x} after the second term.

Resolving Collisions

The methods for dealing with collisions can be classified into the two main types

- ▶ **open addressing**, aka. closed hashing
- ▶ **hashing with chaining**, aka. closed addressing, open hashing.

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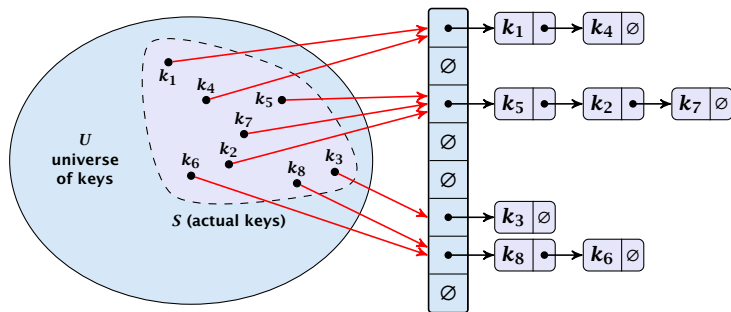
- ▶ **open addressing**, aka. closed hashing
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Hashing with Chaining

Arrange elements that map to the same position in a linear list.

- ▶ Access: compute $h(x)$ and search list for $\text{key}[x]$.
- ▶ Insert: insert at the front of the list.



Hashing with Chaining

Let A denote a strategy for resolving collisions. We use the following notation:

- ▶ A^+ denotes the average time for a **successful** search when using A ;
- ▶ A^- denotes the average time for an **unsuccessful** search when using A ;
- ▶ We parameterize the complexity results in terms of $\alpha := \frac{m}{n}$, the so-called **fill factor** of the hash-table.

We assume **uniform hashing** for the following analysis.

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$$A^- = 1 + \alpha .$$

Hashing with Chaining

For a successful search observe that we do **not** choose a list at random, but we consider a random key k in the hash-table and ask for the search-time for k .

This is 1 plus the number of elements that lie before k in k 's list.

Let k_ℓ denote the ℓ -th key inserted into the table.

Let for two keys k_i and k_j , X_{ij} denote the indicator variable for the event that k_i and k_j hash to the same position. Clearly, $\Pr[X_{ij} = 1] = 1/n$ for uniform hashing.

The expected successful search cost is

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Hence, the expected cost for a successful search is $A^+ \leq 1 + \frac{\alpha}{2}$.

Hashing with Chaining

Disadvantages:

- ▶ pointers increase memory requirements
- ▶ pointers may lead to bad cache efficiency

Advantages:

- ▶ no à priori limit on the number of elements
- ▶ deletion can be implemented efficiently
- ▶ by using balanced trees instead of linked list one can also obtain worst-case guarantees.

Open Addressing

All objects are stored in the table itself.

Define a function $h(k, j)$ that determines the table-position to be examined in the j -th step. The values $h(k, 0), \dots, h(k, n - 1)$ must form a permutation of $0, \dots, n - 1$.

Search(k): Try position $h(k, 0)$; if it is empty your search fails; otw. continue with $h(k, 1), h(k, 2), \dots$.

Insert(x): Search until you find an empty slot; insert your element there. If your search reaches $h(k, n - 1)$, and this slot is non-empty then your table is full.

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Choices for $h(k, j)$:

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$$h(k, i) = h(k) + i \bmod n$$

(sometimes: $h(k, i) = h(k) + ci \bmod n$).

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- ▶ Double hashing:

$$h(k, i) = h_1(k) + ih_2(k) \bmod n.$$

For quadratic probing and double hashing one has to ensure that the search covers all positions in the table (i.e., for double hashing $h_2(k)$ must be relatively prime to n (teilerfremd); for quadratic probing c_1 and c_2 have to be chosen carefully).

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- ▶ Disadvantage: **Primary clustering**. Long sequences of occupied table-positions get longer as they have a larger probability to be hit. Furthermore, they can merge forming larger sequences.

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Let L be the method of linear probing for resolving collisions:

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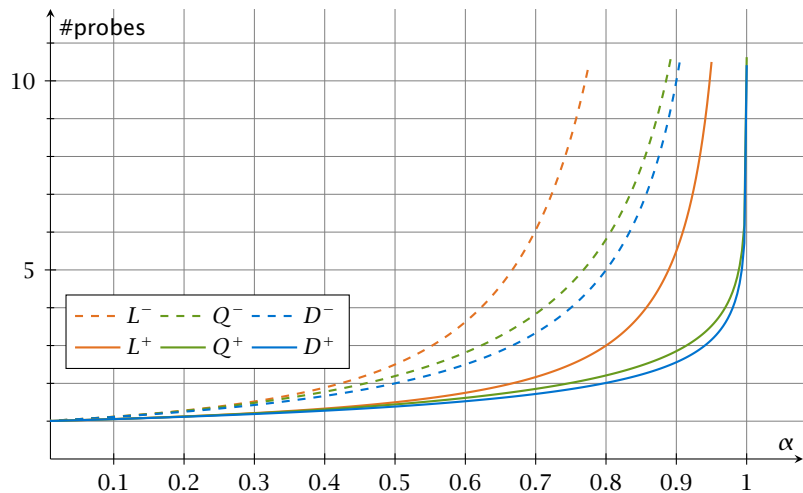
$$D^- \approx \frac{1}{1 - \alpha}$$

Open Addressing

Some values:

α	<i>Linear Probing</i>		<i>Quadratic Probing</i>		<i>Double Hashing</i>	
	L^+	L^-	Q^+	Q^-	D^+	D^-
0.5	1.5	2.5	1.44	2.19	1.39	2
0.9	5.5	50.5	2.85	11.40	2.55	10
0.95	10.5	200.5	3.52	22.05	3.15	20

Open Addressing



Analysis of Idealized Open Address Hashing

We analyze the time for a search in a very idealized Open Addressing scheme.

- ▶ The probe sequence $h(k, 0), h(k, 1), h(k, 2), \dots$ is equally likely to be any permutation of $\langle 0, 1, \dots, n - 1 \rangle$.

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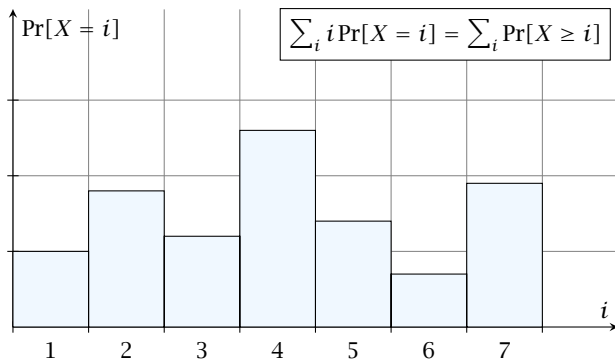
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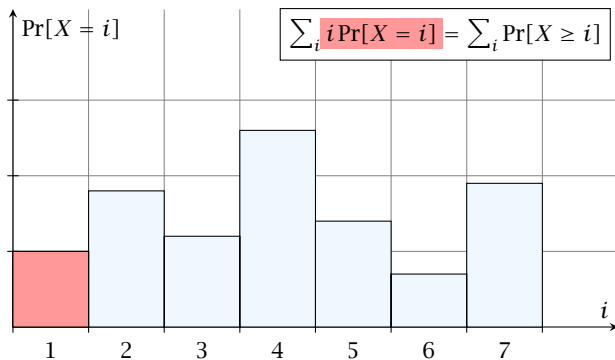
$$\frac{1}{1-\alpha} = 1 + \alpha + \alpha^2 + \alpha^3 + \dots$$

Analysis of Idealized Open Address Hashing



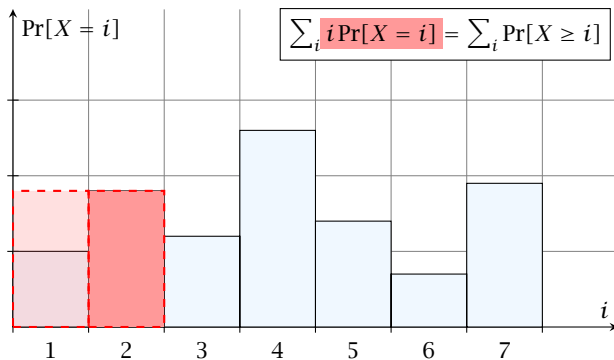
Analysis of Idealized Open Address Hashing

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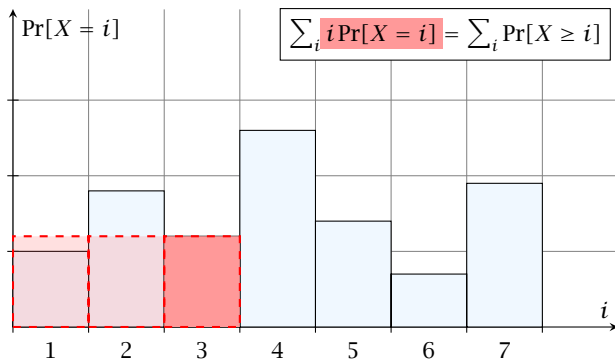
Analysis of Idealized Open Address Hashing

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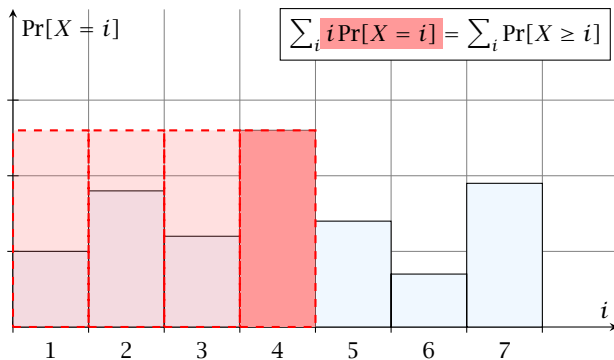
Analysis of Idealized Open Address Hashing

$i = 3$



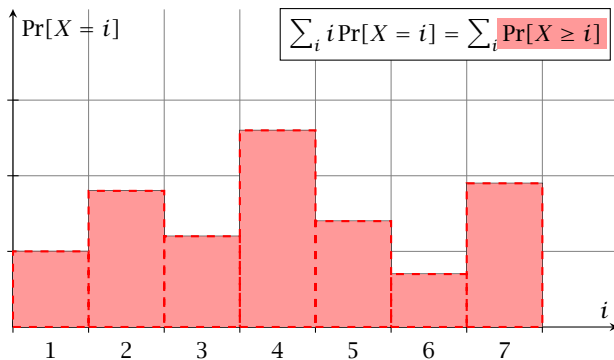
Analysis of Idealized Open Address Hashing

$i = 4$



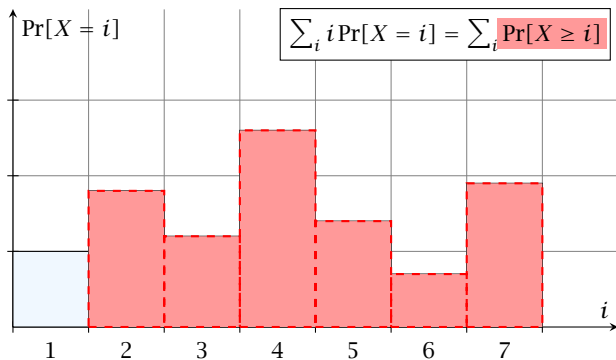
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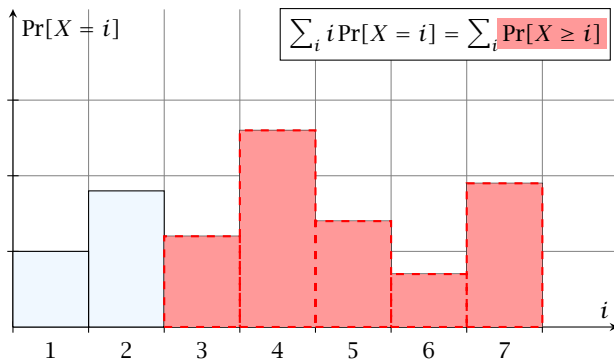
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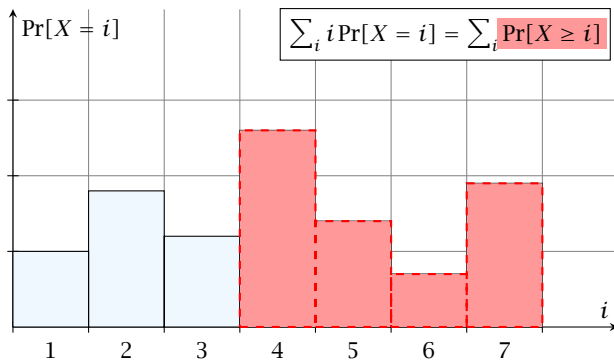
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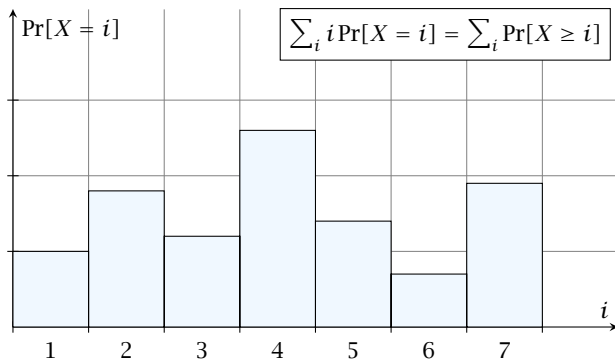


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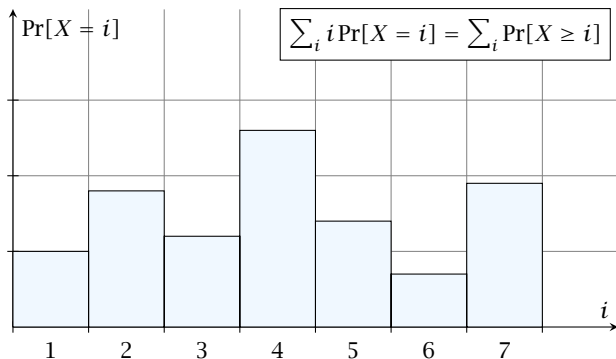
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Analysis of Idealized Open Address Hashing



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The j -th rectangle appears in both sums j times. (j times in the first due to multiplication with j ; and j times in the second for summands $i = 1, 2, \dots, j$)

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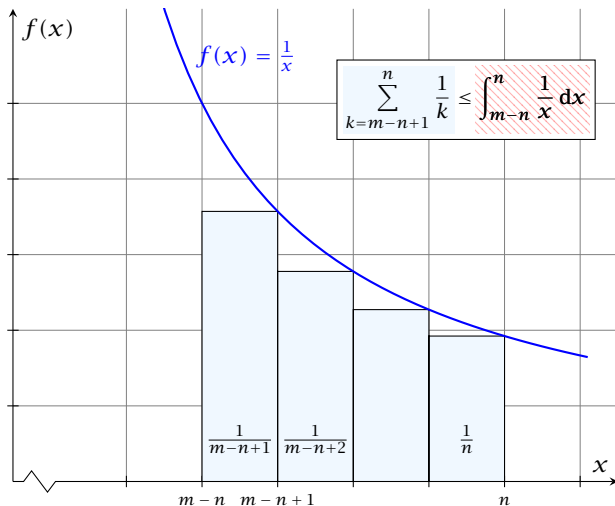
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Deletions in Hashtables

How do we delete in a hash-table?

- ▶ For hashing with chaining this is not a problem. Simply search for the key, and delete the item in the corresponding list.
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- ▶ Simply removing a key might interrupt the probe sequence of other keys which then cannot be found anymore.
- ▶ One can delete an element by replacing it with a **deleted-marker**.
 - ▶ Deleted markers interrupt the probe sequence of other keys which then cannot be found anymore.
 - ▶ Deleted markers can be ignored during insertions.
 - ▶ Deleted markers can be removed during deletions.
 - ▶ Deleted markers can be converted to empty slots during a rehash.
- ▶ The table could fill up with deleted-markers leading to bad performance.
- ▶ If a table contains many deleted-markers (linear fraction of the keys) one can rehash the whole table and amortize the cost for this rehash against the cost for the deletions.

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Algorithm 16 delete(p)

```
1:  $T[p] \leftarrow \text{null}$ 
2:  $p \leftarrow \text{succ}(p)$ 
3: while  $T[p] \neq \text{null}$  do
4:    $y \leftarrow T[p]$ 
5:    $T[p] \leftarrow \text{null}$ 
6:    $p \leftarrow \text{succ}(p)$ 
7:   insert( $y$ )
```

p is the index into the table-cell that contains the object to be deleted.

Pointers into the hash-table become invalid.

Deletions for Linear Probing

Algorithm 16 delete(p)

```
1:  $T[p] \leftarrow \text{null}$ 
2:  $p \leftarrow \text{succ}(p)$ 
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Regardless, of the choice of hash-function there is always an input (a set of keys) that has a very poor worst-case behaviour.

Therefore, so far we assumed that the hash-function is random so that regardless of the input the average case behaviour is good.

However, the assumption of uniform hashing that h is chosen randomly from all functions $f: U \rightarrow [0, \dots, n-1]$ is clearly unrealistic as there are $n^{|U|}$ such functions. Even writing down such a function would take $|U| \log n$ bits.

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Definition 5

A class \mathcal{H} of hash-functions from the universe U into the set $\{0, \dots, n-1\}$ is called **universal** if for all $u_1, u_2 \in U$ with $u_1 \neq u_2$

$$\Pr[h(u_1) = h(u_2)] \leq \frac{1}{n} ,$$

where the probability is w. r. t. the choice of a random hash-function from set \mathcal{H} .

Note that this means that the probability of a collision between two arbitrary elements is at most $\frac{1}{n}$.

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Definition 6

A class \mathcal{H} of hash-functions from the universe U into the set $\{0, \dots, n-1\}$ is called **2-independent** (pairwise independent) if the following two conditions hold

- ▶ For any key $u \in U$, and $t \in \{0, \dots, n-1\}$ $\Pr[h(u) = t] = \frac{1}{n}$, i.e., a key is distributed uniformly within the hash-table.
- ▶ For all $u_1, u_2 \in U$ with $u_1 \neq u_2$, and for any two hash-positions t_1, t_2 :

$$\Pr[h(u_1) = t_1 \wedge h(u_2) = t_2] \leq \frac{1}{n^2} .$$

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A class \mathcal{H} of hash-functions from the universe U into the set $\{0, \dots, n-1\}$ is called **k -independent** if for any choice of $\ell \leq k$ distinct keys $u_1, \dots, u_\ell \in U$, and for any set of ℓ not necessarily distinct hash-positions t_1, \dots, t_ℓ :

$$\Pr[h(u_1) = t_1 \wedge \dots \wedge h(u_\ell) = t_\ell] \leq \frac{1}{n^\ell} ,$$

where the probability is w. r. t. the choice of a random hash-function from set \mathcal{H} .

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Definition 8

A class \mathcal{H} of hash-functions from the universe U into the set $\{0, \dots, n-1\}$ is called (μ, k) -independent if for any choice of $\ell \leq k$ distinct keys $u_1, \dots, u_\ell \in U$, and for any set of ℓ not necessarily distinct hash-positions t_1, \dots, t_ℓ :

$$\Pr[h(u_1) = t_1 \wedge \dots \wedge h(u_\ell) = t_\ell] \leq \frac{\mu}{n^\ell},$$

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Universal Hashing

Let $U := \{0, \dots, p-1\}$ for a prime p . Let $\mathbb{Z}_p := \{0, \dots, p-1\}$, and let $\mathbb{Z}_p^* := \{1, \dots, p-1\}$ denote the set of invertible elements in \mathbb{Z}_p .

Define

$$h_{a,b}(x) := (ax + b \bmod p) \bmod n$$

Lemma 9

The class

$$\mathcal{H} = \{h_{a,b} \mid a \in \mathbb{Z}_p^*, b \in \mathbb{Z}_p\}$$

is a universal class of hash-functions from U to $\{0, \dots, n-1\}$.

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Proof.

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$$\triangleright ax + b \not\equiv ay + b \pmod{p}$$

if $a \neq 0$, then a^{-1} exists and we can multiply both sides with a^{-1} .

Multiplying with a^{-1} gives $x \not\equiv y \pmod{p}$.

Since $x \neq y$, we have $x \not\equiv y \pmod{p}$.

Therefore, the probability of a collision is $1/n$.

Q.E.D.

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Let $x, y \in U$ be two distinct keys. We have to show that the probability of a collision is only $1/n$.

$$\blacktriangleright ax + b \not\equiv ay + b \pmod{p}$$

If $x \neq y$ then $(x - y) \not\equiv 0 \pmod{p}$.

Multiplying with $a \not\equiv 0 \pmod{p}$ gives

$$a(x - y) \not\equiv 0 \pmod{p}$$

where we use that \mathbb{Z}_p is a field (Körper) and, hence, has no zero divisors (nullteilerfrei).

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- ▶ The hash-function does not generate collisions before the $(\text{mod } n)$ -operation. Furthermore, every choice (a, b) is mapped to a different pair (t_x, t_y) with $t_x := ax + b$ and $t_y := ay + b$.

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$$t_y \equiv ay + b \pmod{p}$$

$$a \equiv (t_x - t_y)(x - y)^{-1} \pmod{p}$$

$$b \equiv t_y - ay \pmod{p}$$

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There is a one-to-one correspondence between hash-functions (pairs (a, b) , $a \neq 0$) and pairs (t_x, t_y) , $t_x \neq t_y$.

Therefore, we can view the first step (before the $\text{mod } n$ -operation) as choosing a pair (t_x, t_y) , $t_x \neq t_y$ uniformly at random.

What happens when we do the $\text{mod } n$ operation?

Fix a value t_x . There are $p - 1$ possible values for choosing t_y .

From the range $0, \dots, p - 1$ the values $t_x, t_x + n, t_x + 2n, \dots$ map to t_x after the modulo-operation. These are at most $\lceil p/n \rceil$ values.

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As $t_y \neq t_x$ there are

possibilities for choosing t_y such that the final hash-value creates a collision.

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It is also possible to show that \mathcal{H} is an (almost) pairwise independent class of hash-functions.

$$\Pr_{t_x \neq t_y \in \mathbb{Z}_p^2} \left[\begin{array}{l} t_x \bmod n = h_1 \\ t_y \bmod n = h_2 \end{array} \right]$$

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$$\frac{\lfloor \frac{p}{n} \rfloor^2}{p(p-1)} \leq \Pr_{t_x \neq t_y \in \mathbb{Z}_p^2} \left[\begin{array}{l} t_x \bmod n = h_1 \\ t_y \bmod n = h_2 \end{array} \right] \leq \frac{\lfloor \frac{p}{n} \rfloor^2}{p(p-1)}$$

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Note that the middle is the probability that $h(x) = h_1$ and $h(y) = h_2$. The total number of choices for (t_x, t_y) is $p(p-1)$. The number of choices for t_x (t_y) such that $t_x \bmod n = h_1$ ($t_y \bmod n = h_2$) lies between $\lfloor \frac{p}{n} \rfloor$ and $\lceil \frac{p}{n} \rceil$.

Universal Hashing

Definition 10

Let $d \in \mathbb{N}$; $q \geq (d + 1)n$ be a prime; and let $\bar{a} \in \{0, \dots, q - 1\}^{d+1}$. Define for $x \in \{0, \dots, q - 1\}$

$$h_{\bar{a}}(x) := \left(\sum_{i=0}^d a_i x^i \bmod q \right) \bmod n .$$

Let $\mathcal{H}_n^d := \{h_{\bar{a}} \mid \bar{a} \in \{0, \dots, q - 1\}^{d+1}\}$. The class \mathcal{H}_n^d is $(e, d + 1)$ -independent.

Note that in the previous case we had $d = 1$ and chose $a_d \neq 0$.

Universal Hashing

For the coefficients $\bar{a} \in \{0, \dots, q-1\}^{d+1}$ let $f_{\bar{a}}$ denote the polynomial

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Fix $\ell \leq d + 1$; let $x_1, \dots, x_\ell \in \{0, \dots, q - 1\}$ be keys, and let t_1, \dots, t_ℓ denote the corresponding hash-function values.

Let $A^\ell = \{h_{\bar{a}} \in \mathcal{H} \mid h_{\bar{a}}(x_i) = t_i \text{ for all } i \in \{1, \dots, \ell\}\}$

Then

$$h_{\bar{a}} \in A^\ell \Leftrightarrow h_{\bar{a}} = f_{\bar{a}} \bmod n \text{ and}$$

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In order to obtain the cardinality of A^ℓ we choose our polynomial by fixing $d + 1$ points.

We first fix the values for inputs x_1, \dots, x_ℓ .

We have

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In order to obtain the cardinality of A^ℓ we choose our polynomial by fixing $d + 1$ points.

We first fix the values for inputs x_1, \dots, x_ℓ .

We have

$$|B_1| \cdot \dots \cdot |B_\ell|$$

possibilities to do this (so that $h_{\bar{a}}(x_i) = t_i$).

Universal Hashing

Fix $\ell \leq d + 1$; let $x_1, \dots, x_\ell \in \{0, \dots, q - 1\}$ be keys, and let t_1, \dots, t_ℓ denote the corresponding hash-function values.

Let $A^\ell = \{h_{\bar{a}} \in \mathcal{H} \mid h_{\bar{a}}(x_i) = t_i \text{ for all } i \in \{1, \dots, \ell\}\}$

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Now, we choose $d - \ell + 1$ other inputs and choose their value arbitrarily. We have $q^{d-\ell+1}$ possibilities to do this.

Therefore we have

$$|B_1| \cdot \dots \cdot |B_\ell| \cdot q^{d-\ell+1} \leq \left\lceil \frac{q}{n} \right\rceil^\ell \cdot q^{d-\ell+1}$$

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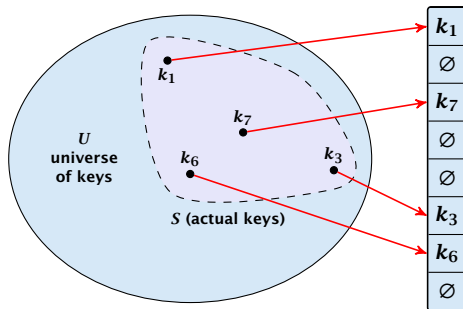
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This shows that the \mathcal{H} is $(e, d+1)$ -universal.

The last step followed from $q \geq (d+1)n$, and $\ell \leq d+1$.

Perfect Hashing

Suppose that we **know** the set S of actual keys (no insert/no delete). Then we may want to design a **simple** hash-function that maps all these keys to different memory locations.



Perfect Hashing

Let $m = |S|$. We could simply choose the hash-table size very large so that we don't get any collisions.

Using a universal hash-function the expected number of collisions is

$$E[\#\text{Collisions}] = \binom{m}{2} \cdot \frac{1}{n}.$$

If we choose $n = m^2$ the expected number of collisions is strictly less than $\frac{1}{2}$.

Can we get an upper bound on the probability of having collisions?

The probability of having 1 or more collisions can be at most $\frac{1}{2}$ as otherwise the expectation would be larger than $\frac{1}{2}$.

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We can find such a hash-function by a few trials.

However, a hash-table size of $n = m^2$ is very very high.

We construct a two-level scheme. We first use a hash-function that maps elements from S to m buckets.

Let m_j denote the number of items that are hashed to the j -th bucket. For each bucket we choose a second hash-function that maps the elements of the bucket into a table of size m_j^2 . The second function can be chosen such that all elements are mapped to different locations.

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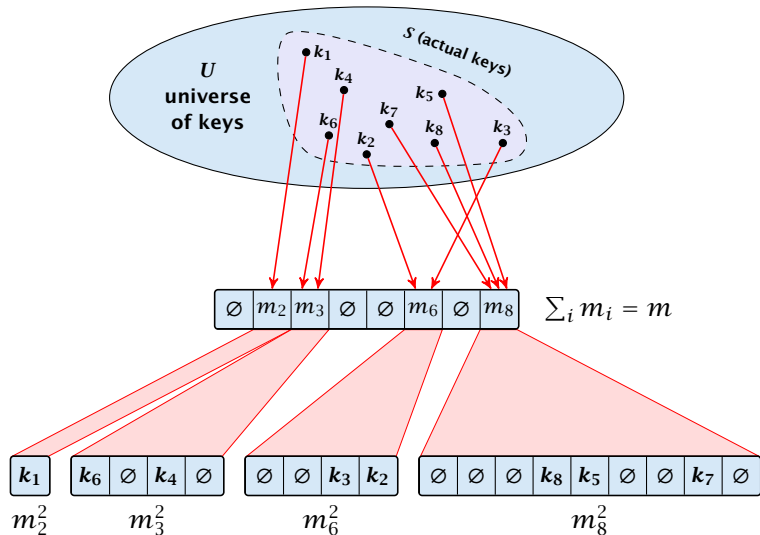
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$$= 2 \binom{m}{2} \frac{1}{m} + m = 2m - 1 .$$

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We need only $\mathcal{O}(m)$ time to construct a hash-function h with $\sum_j m_j^2 = \mathcal{O}(4m)$, because with probability at least $1/2$ a random function from a universal family will have this property.

Then we construct a hash-table h_j for every bucket. This takes expected time $\mathcal{O}(m_j)$ for every bucket. A random function h_j is collision-free with probability at least $1/2$. We need $\mathcal{O}(m_j)$ to test this.

We only need that the hash-functions are chosen from a universal family!!!

Cuckoo Hashing

Goal:

Try to generate a hash-table with constant worst-case search time in a dynamic scenario.

Two hash-tables T_1 and T_2 of size m , with hash functions h_1 and h_2 .

An object x is either stored at location $T_1[h_1(x)]$ or $T_2[h_2(x)]$.

Search clearly takes constant time if the above conditions are met.

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Try to generate a hash-table with constant worst-case search time in a dynamic scenario.

- ▶ Two hash-tables $T_1[0, \dots, n-1]$ and $T_2[0, \dots, n-1]$, with hash-functions h_1 , and h_2 .
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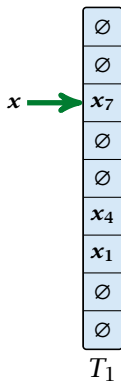
T_1



T_2

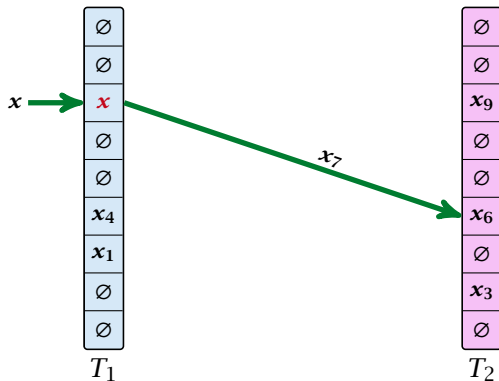
Cuckoo Hashing

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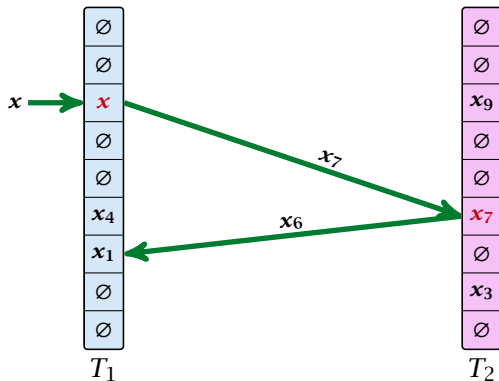
Cuckoo Hashing

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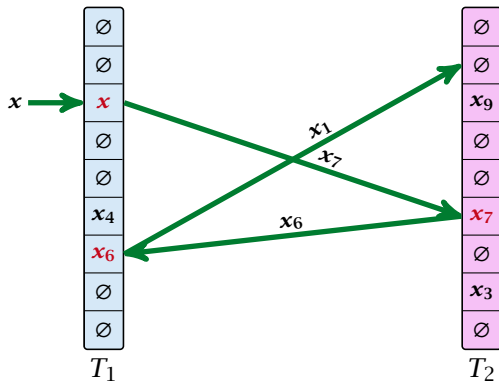
Cuckoo Hashing

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Cuckoo Hashing

Algorithm 17 Cuckoo-Insert(x)

```
1: if  $T_1[h_1(x)] = x \vee T_2[h_2(x)] = x$  then return  
2: steps  $\leftarrow$  1  
3: while steps  $\leq$  maxsteps do  
4:     exchange  $x$  and  $T_1[h_1(x)]$   
5:     if  $x = \text{null}$  then return  
6:     exchange  $x$  and  $T_2[h_2(x)]$   
7:     if  $x = \text{null}$  then return  
8:     steps  $\leftarrow$  steps + 1  
9: rehash() // change hash-functions; rehash everything  
10: Cuckoo-Insert( $x$ )
```

Cuckoo Hashing

- ▶ We call one iteration through the while-loop a **step** of the algorithm.
- ▶ We call a sequence of iterations through the while-loop without the termination condition becoming true a **phase** of the algorithm.
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What is the expected time for an insert-operation?

We first analyze the probability that we end-up in an infinite loop (that is then terminated after `maxsteps` steps).

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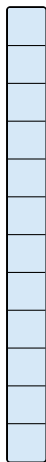
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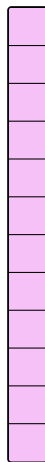
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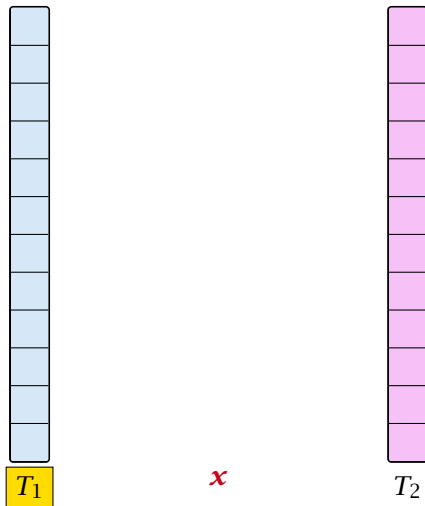


T_1

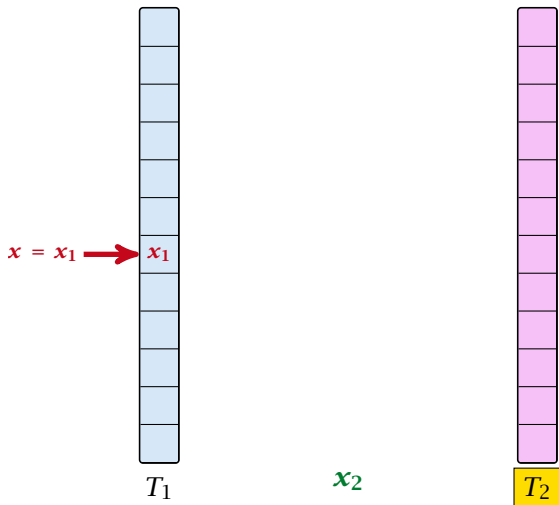


T_2

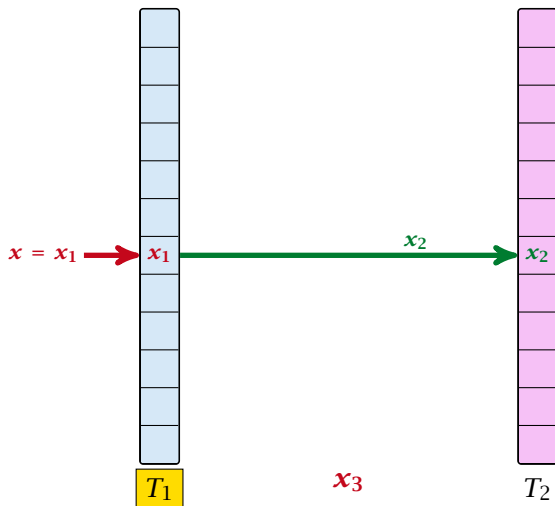
Cuckoo Hashing: Insert



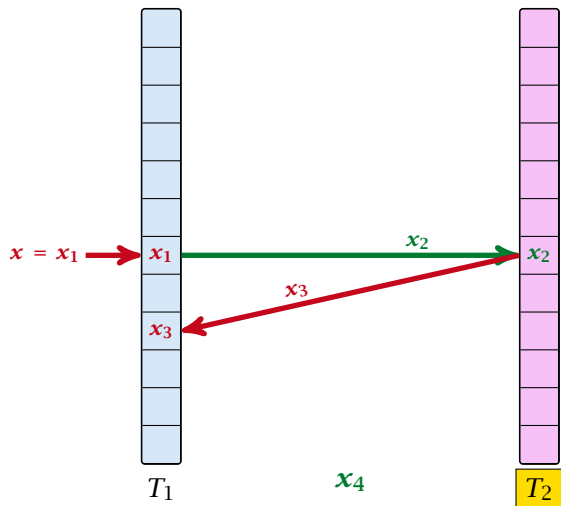
Cuckoo Hashing: Insert



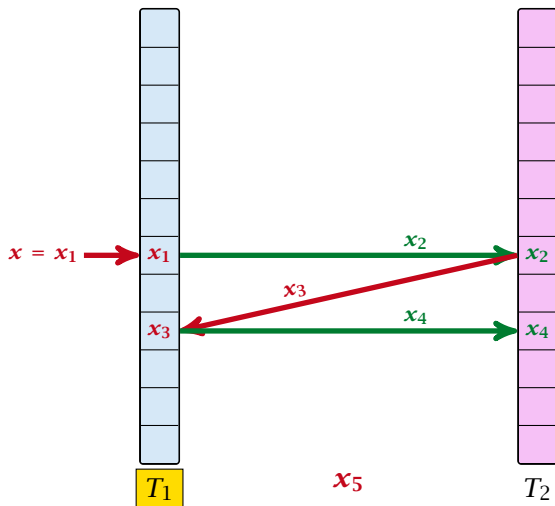
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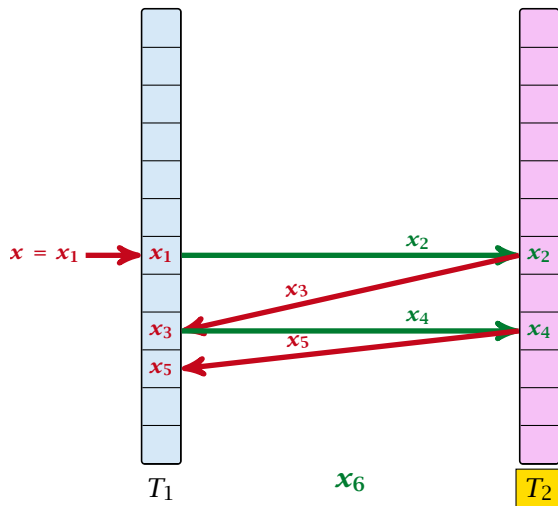
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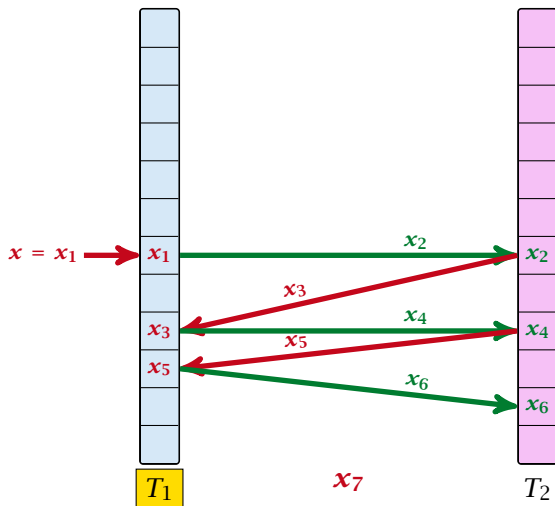
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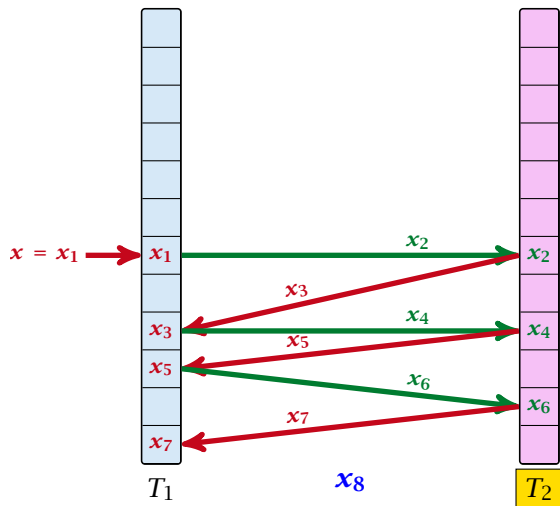
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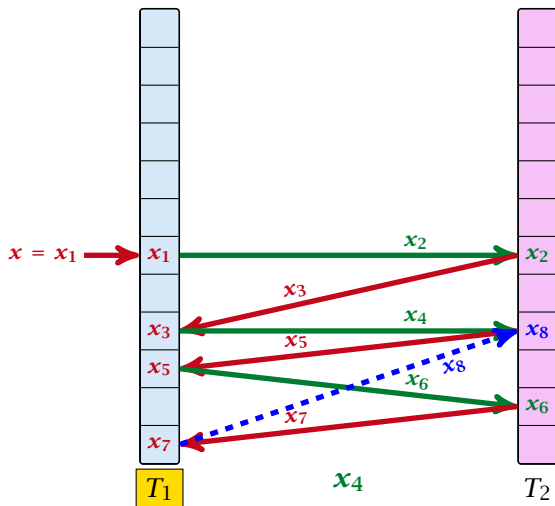
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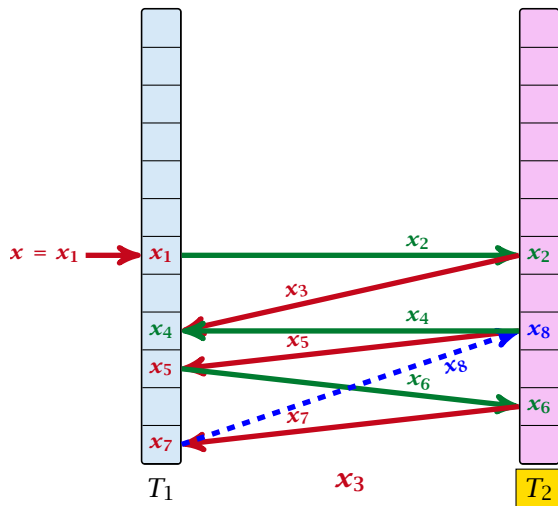
Cuckoo Hashing: Insert



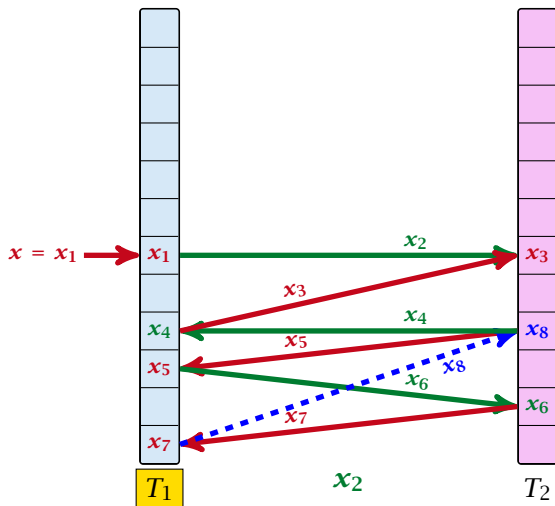
Cuckoo Hashing: Insert



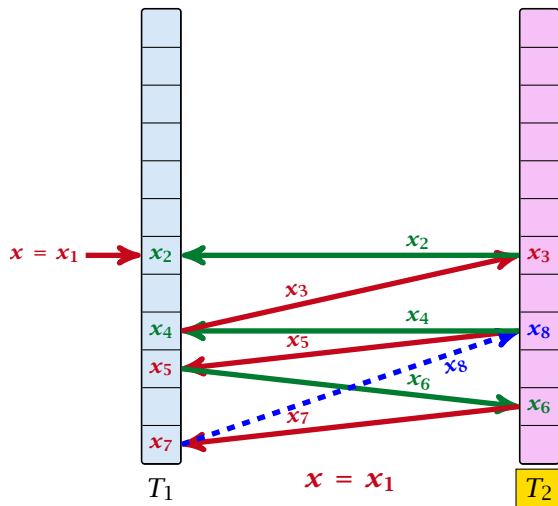
Cuckoo Hashing: Insert



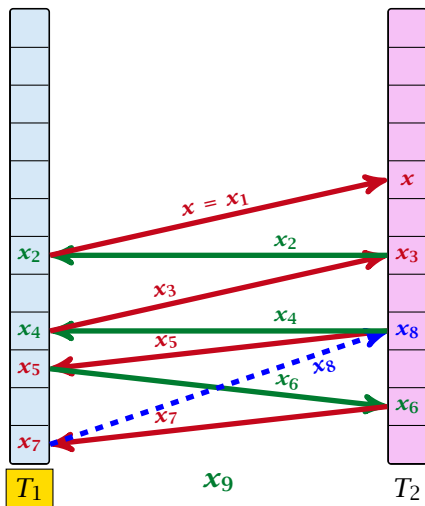
Cuckoo Hashing: Insert



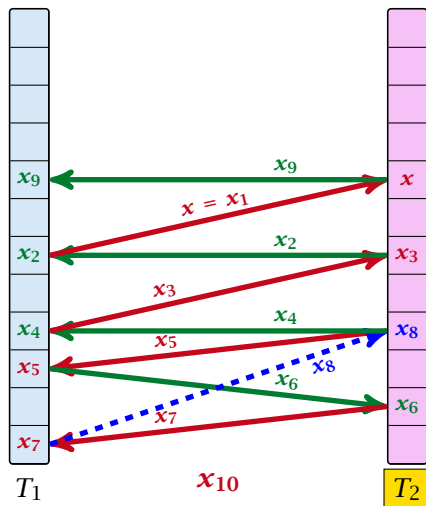
Cuckoo Hashing: Insert



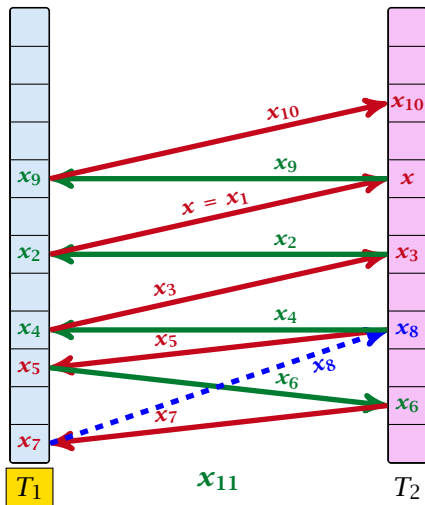
Cuckoo Hashing: Insert



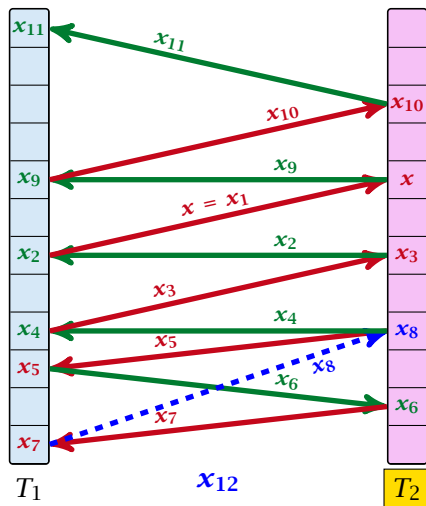
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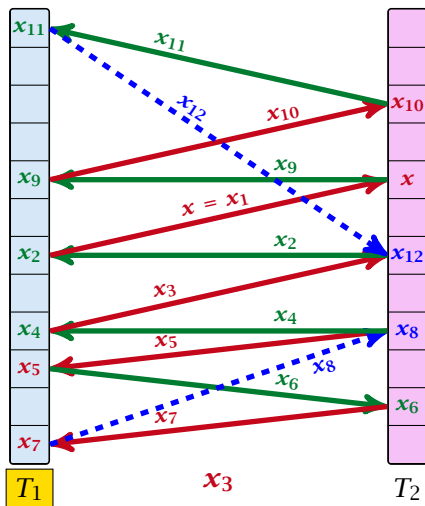
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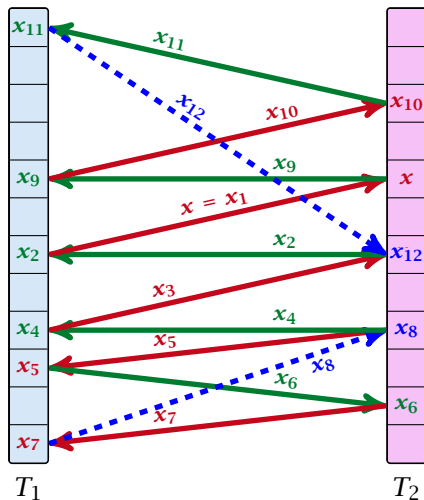
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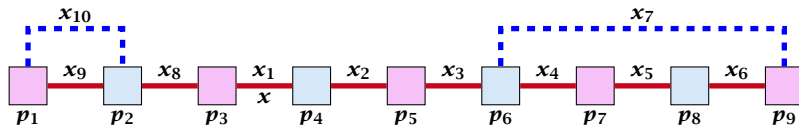
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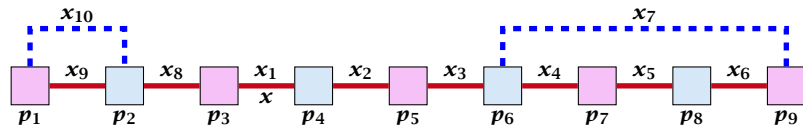


Cuckoo Hashing



A cycle-structure of size s is defined by

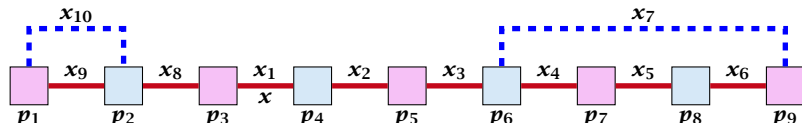
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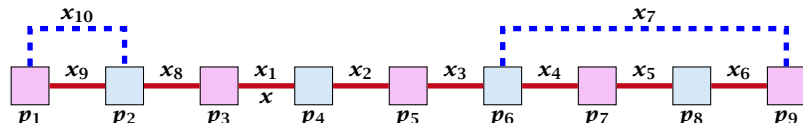
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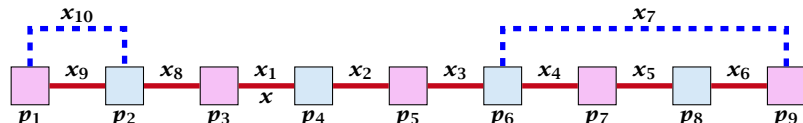
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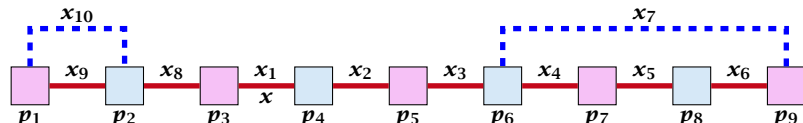
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Cuckoo Hashing

A cycle-structure is **active** if for every key x_ℓ (linking a cell p_i from T_1 and a cell p_j from T_2) we have

$$h_1(x_\ell) = p_i \quad \text{and} \quad h_2(x_\ell) = p_j$$

Observation:

If during a phase the insert-procedure runs into a cycle there must exist an active cycle structure of size $s \geq 3$.

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Observation:

If during a phase the insert-procedure runs into a cycle there must exist an active cycle structure of size $s \geq 3$.

Cuckoo Hashing

What is the probability that all keys in a cycle-structure of size s correctly map into their T_1 -cell?

This probability is at most $\frac{\mu}{n^s}$ since h_1 is a (μ, s) -independent hash-function.

What is the probability that all keys in the cycle-structure of size s correctly map into their T_2 -cell?

This probability is at most $\frac{\mu}{n^s}$ since h_2 is a (μ, s) -independent hash-function.

These events are independent.

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The probability that a given cycle-structure of size s is active is at most $\frac{\mu^2}{n^{2s}}$.

What is the probability that there exists an active cycle structure of size s ?

Cuckoo Hashing

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What is the probability that **there exists** an active cycle structure of size s ?

Cuckoo Hashing

The number of cycle-structures of size s is at most

$$s^3 \cdot n^{s-1} \cdot m^{s-1} .$$

Theorem 11.11 (see [11]) When treated as a directed graph, cuckoo hashing has only forward and backward links.

There are at most s possibilities to choose where to place

the s keys. There are s possibilities to choose the key that

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Cuckoo Hashing

The number of cycle-structures of size s is at most

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- ▶ There are at most s^2 possibilities where to attach the forward and backward links.
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Cuckoo Hashing

The probability that there exists an active cycle-structure is therefore at most

$$\sum_{s=3}^{\infty} s^3 \cdot n^{s-1} \cdot m^{s-1} \cdot \frac{\mu^2}{n^{2s}}$$

Cuckoo Hashing

The probability that there exists an active cycle-structure is therefore at most

$$\sum_{s=3}^{\infty} s^3 \cdot n^{s-1} \cdot m^{s-1} \cdot \frac{\mu^2}{n^{2s}} = \frac{\mu^2}{nm} \sum_{s=3}^{\infty} s^3 \left(\frac{m}{n}\right)^s$$

Cuckoo Hashing

The probability that there exists an active cycle-structure is therefore at most

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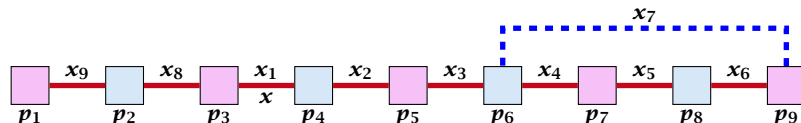
Hence,

$$\Pr[\text{cycle}] = \mathcal{O}\left(\frac{1}{m^2}\right).$$

Cuckoo Hashing

Now, we analyze the probability that a phase is not successful without running into a closed cycle.

Cuckoo Hashing



Sequence of visited keys:

$x = x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_3, x_2, x_1 = x, x_8, x_9, \dots$

Cuckoo Hashing

Consider the sequence of not necessarily distinct keys starting with x in the order that they are visited during the phase.

Lemma 11

If the sequence is of length p then there exists a sub-sequence of at least $\frac{p+2}{3}$ keys starting with x of distinct keys.

Cuckoo Hashing

Consider the sequence of not necessarily distinct keys starting with x in the order that they are visited during the phase.

Lemma 11

*If the sequence is of length p then there exists a sub-sequence of at least $\frac{p+2}{3}$ keys starting with x of *distinct* keys.*

Cuckoo Hashing

Proof.

Let i be the number of keys (including x) that we see before the first repeated key. Let j denote the total number of distinct keys.

The sequence is of the form:

$$x = x_1 \rightarrow x_2 \rightarrow \dots \rightarrow x_i \rightarrow x_r \rightarrow x_{r-1} \rightarrow \dots \rightarrow x_1 \rightarrow x_{i+1} \rightarrow \dots \rightarrow x_j$$

As $r \leq i - 1$ the length p of the sequence is

$$p = i + r + (j - i) \leq i + j - 1 .$$

Either sub-sequence $x_1 \rightarrow x_2 \rightarrow \dots \rightarrow x_i$ or sub-sequence $x_1 \rightarrow x_{i+1} \rightarrow \dots \rightarrow x_j$ has at least $\frac{p+2}{3}$ elements. □

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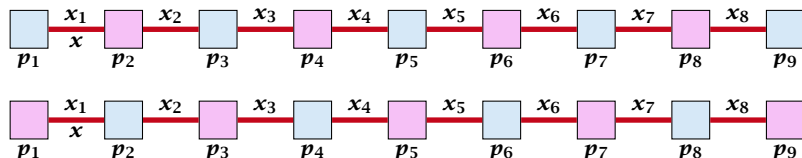
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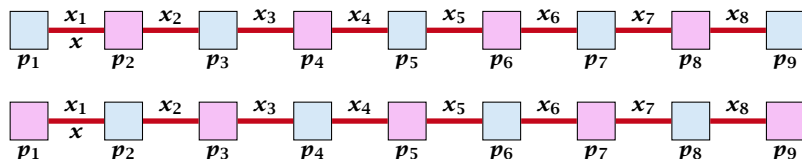
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Cuckoo Hashing



A path-structure of size s is defined by

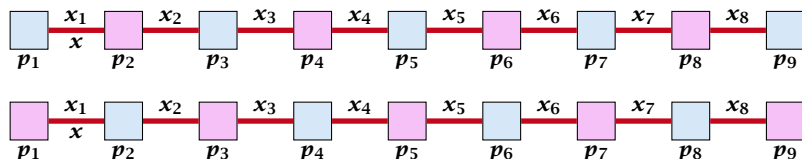
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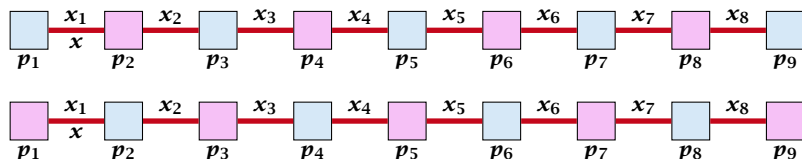
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$$h_1(x_\ell) = p_i \quad \text{and} \quad h_2(x_\ell) = p_j$$

Observation:

If a phase takes at least t steps without running into a cycle there must exist an active path-structure of size $(2t + 2)/3$.

Cuckoo Hashing

The probability that a given path-structure of size s is active is at most $\frac{\mu^2}{n^{2s}}$.

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by choosing $\ell \geq \log \left(\frac{1}{2\mu^2 m^2} \right) / \log \left(\frac{1}{1+\epsilon} \right) = \log (2\mu^2 m^2) / \log (1 + \epsilon)$

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$$\begin{aligned} & \Pr[\text{unsuccessful} \mid \text{no cycle}] \\ & \leq \Pr[\exists \text{ active path-structure of size at least } \frac{2\text{maxsteps}-1}{3}] \\ & \leq \Pr[\exists \text{ active path-structure of size at least } \ell + 1] \\ & \leq \Pr[\exists \text{ active path-structure of size exactly } \ell + 1] \\ & \leq 2\mu^2 \left(\frac{1}{1+\epsilon}\right)^\ell \leq \frac{1}{m^2} \end{aligned}$$

by choosing $\ell \geq \log\left(\frac{1}{2\mu^2 m^2}\right) / \log\left(\frac{1}{1+\epsilon}\right) = \log(2\mu^2 m^2) / \log(1+\epsilon)$

This gives $\text{maxsteps} = \Theta(\log m)$.

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This means the expected cost for a successful phase is constant (even after accounting for the cost of the incomplete step that finishes the phase).

Cuckoo Hashing

A phase that is not successful induces cost for doing a complete rehash (this dominates the cost for the steps in the phase).

The probability that a phase is not successful is $p = \mathcal{O}(1/m^2)$ (probability $\mathcal{O}(1/m^2)$ of running into a cycle and probability $\mathcal{O}(1/m^2)$ of reaching maxsteps without running into a cycle).

A rehash try requires m insertions and takes expected constant time per insertion. It fails with probability $p := \mathcal{O}(1/m)$.

The expected number of unsuccessful rehashes is

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The expected cost for all rehashes is

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What kind of hash-functions do we need?

Since maxsteps is $\Theta(\log m)$ the largest size of a path-structure or cycle-structure contains just $\Theta(\log m)$ different keys.

Therefore, it is sufficient to have $(\mu, \Theta(\log m))$ -independent hash-functions.

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How do we make sure that $n \geq (1 + \epsilon)m$?

- ▶ Let $\alpha := 1/(1 + \epsilon)$.
- ▶ Keep track of the number of elements in the table. When $m \geq \alpha n$ we double n and do a complete re-hash (table-expand).
- ▶ Whenever m drops below $\alpha n/4$ we divide n by 2 and do a rehash (table-shrink).
- ▶ Note that right after a change in table-size we have $m = \alpha n/2$. In order for a table-expand to occur at least $\alpha n/2$ insertions are required. Similar, for a table-shrink at least $\alpha n/4$ deletions must occur.
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Lemma 12

Cuckoo Hashing has an expected constant insert-time and a worst-case constant search-time.

Note that the above lemma only holds if the fill-factor (number of keys/total number of hash-table slots) is at most $\frac{1}{2(1+\epsilon)}$.

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