4 Simplex Algorithm

Enumerating all basic feasible solutions (BFS), in order to find the optimum is slow.

Simplex Algorithm [George Dantzig 1947]
Move from BFS to adjacent BFS, without decreasing objective function.

Two BFSs are called adjacent if the bases just differ in one variable.

4 Simplex Algorithm

$$\begin{array}{lrcrcccccl}
\max & 13a & + & 23b & & & & & & \\
\text{s.t.} & 5a & + & 15b & + s_c & & & & = & 480 \\
 & 4a & + & 4b & & + s_h & & & = & 160 \\
 & 35a & + & 20b & & & + s_m & & = & 1190 \\
 & a & , & b & , s_c & , s_h & , s_m & & \geq & 0
\end{array}$$

$$\begin{array}{lrcrcccccl}
\max Z & & & & & & & & & \\
 & 13a & + & 23b & & & & - Z & = & 0 \\
 & 5a & + & 15b & + s_c & & & & = & 480 \\
 & 4a & + & 4b & & + s_h & & & = & 160 \\
 & 35a & + & 20b & & & + s_m & & = & 1190 \\
 & a & , & b & , s_c & , s_h & , s_m & & \geq & 0
\end{array}$$

basis $= \{s_c, s_h, s_m\}$

$A = B = 0$

$Z = 0$

$s_c = 480$

$s_h = 160$

$s_m = 1190$

Pivoting Step

$$
\begin{array}{l}
\max\ Z \\
\quad \begin{array}{rcrcccccccccl}
13a & + & 23b & & & & & & & - & Z & = & 0 \\
5a & + & 15b & + & s_c & & & & & & & = & 480 \\
4a & + & 4b & & & + & s_h & & & & & = & 160 \\
35a & + & 20b & & & & & + & s_m & & & = & 1190 \\
a & , & b & , & s_c & , & s_h & , & s_m & & & \geq & 0
\end{array}
\end{array}
$$

$$
\begin{array}{lcl}
\text{basis} & = & \{s_c, s_h, s_m\} \\
a & = & b = 0 \\
Z & = & 0 \\
s_c & = & 480 \\
s_h & = & 160 \\
s_m & = & 1190
\end{array}
$$

- choose variable to bring into the basis
- chosen variable should have positive coefficient in objective function
- apply min-ratio test to find out by how much the variable can be increased
- pivot on row found by min-ratio test
- the existing basis variable in this row leaves the basis

$$
\begin{array}{l}
\max\ Z \\
\begin{array}{rcrcccccl}
13a & + & 23\boldsymbol{b} & & & & & -\ Z & = 0 \\
5a & + & 15\boldsymbol{b} & + s_c & & & & & = 480 \\
4a & + & 4\boldsymbol{b} & & + s_h & & & & = 160 \\
35a & + & 20\boldsymbol{b} & & & + s_m & & & = 1190 \\
a & , & \boldsymbol{b} & ,\ s_c & ,\ s_h & ,\ s_m & & & \geq 0
\end{array}
\end{array}
$$

$$
\begin{array}{l}
\text{basis} = \{s_c, s_h, s_m\} \\
a = b = 0 \\
Z = 0 \\
s_c = 480 \\
s_h = 160 \\
s_m = 1190
\end{array}
$$

- Choose variable with coefficient ≥ 0 as entering variable.
- If we keep $a = 0$ and increase b from 0 to $\theta > 0$ s.t. all constraints ($Ax = b, x \geq 0$) are still fulfilled the objective value Z will strictly increase.
- For maintaining $Ax = b$ we need e.g. to set $s_c = 480 - 15\theta$.
- Choosing $\theta = \min\{480/15, 160/4, 1190/20\}$ ensures that in the new solution one current basic variable becomes 0, and no variable goes negative.
- The basic variable in the row that gives $\min\{480/15, 160/4, 1190/20\}$ becomes the leaving variable.

$$\begin{array}{l} \max Z \\ \begin{array}{rcrcrcrcrcrcl} 13a & + & 23b & & & & & & & - & Z & = & 0 \\ 5a & + & 15b & + & s_c & & & & & & & = & 480 \\ 4a & + & 4b & & & + & s_h & & & & & = & 160 \\ 35a & + & 20b & & & & & + & s_m & & & = & 1190 \\ a & , & b & , & s_c & , & s_h & , & s_m & & & \geq & 0 \end{array} \end{array}$$

$$\begin{array}{l} \text{basis} = \{s_c, s_h, s_m\} \\ a = b = 0 \\ Z = 0 \\ s_c = 480 \\ s_h = 160 \\ s_m = 1190 \end{array}$$

Substitute $b = \frac{1}{15}(480 - 5a - s_c)$.

$$\begin{array}{l} \max Z \\ \begin{array}{rcrcrcrcrcrcl} \frac{16}{3}a & & & - & \frac{23}{15}s_c & & & & & - & Z & = & -736 \\ \frac{1}{3}a & + & b & + & \frac{1}{15}s_c & & & & & & & = & 32 \\ \frac{8}{3}a & & & - & \frac{4}{15}s_c & + & s_h & & & & & = & 32 \\ \frac{85}{3}a & & & - & \frac{4}{3}s_c & & & + & s_m & & & = & 550 \\ a & , & b & , & s_c & , & s_h & , & s_m & & & \geq & 0 \end{array} \end{array}$$

$$\begin{array}{l} \text{basis} = \{b, s_h, s_m\} \\ a = s_c = 0 \\ Z = 736 \\ b = 32 \\ s_h = 32 \\ s_m = 550 \end{array}$$

max Z

$$\begin{array}{rcrcrcrcrcl}
\frac{16}{3}\boldsymbol{a} & & & - & \frac{23}{15}s_c & & & & & -\ Z & = -736 \\
\frac{1}{3}\boldsymbol{a} & + & b & + & \frac{1}{15}s_c & & & & & & = 32 \\
\frac{8}{3}\boldsymbol{a} & & & - & \frac{4}{15}s_c & + & s_h & & & & = 32 \\
\frac{85}{3}\boldsymbol{a} & & & - & \frac{4}{3}s_c & & & + & s_m & & = 550 \\
\boldsymbol{a} & , & b & , & s_c & , & s_h & , & s_m & & \geq 0
\end{array}$$

basis $= \{b, s_h, s_m\}$
$a = s_c = 0$
$Z = 736$

$b = 32$
$s_h = 32$
$s_m = 550$

Choose variable a to bring into basis.

Computing $\min\{3 \cdot 32, {}^{3\cdot 32}/_8, {}^{3\cdot 550}/_{85}\}$ means pivot on line 2.

Substitute $a = \frac{3}{8}(32 + \frac{4}{15}s_c - s_h)$.

max Z

$$\begin{array}{rcrcrcrcrcl}
 & & & - & s_c & - & 2s_h & & & -\ Z & = -800 \\
 & & b & + & \frac{1}{10}s_c & - & \frac{1}{8}s_h & & & & = 28 \\
a & & & - & \frac{1}{10}s_c & + & \frac{3}{8}s_h & & & & = 12 \\
 & & & & \frac{3}{2}s_c & - & \frac{85}{8}s_h & + & s_m & & = 210 \\
a & , & b & , & s_c & , & s_h & , & s_m & & \geq 0
\end{array}$$

basis $= \{a, b, s_m\}$
$s_c = s_h = 0$
$Z = 800$

$b = 28$
$a = 12$
$s_m = 210$

4 Simplex Algorithm

Pivoting stops when all coefficients in the objective function are non-positive.

Solution is optimal:

- any feasible solution satisfies all equations in the tableaux
- in particular: $Z = 800 - s_c - 2s_h$, $s_c \geq 0, s_h \geq 0$
- hence optimum solution value is at most 800
- the current solution has value 800

Matrix View

Let our linear program be

$$\begin{array}{ccccc} c_B^T x_B & + & c_N^T x_N & = & Z \\ A_B x_B & + & A_N x_N & = & b \\ x_B & , & x_N & \geq & 0 \end{array}$$

The simplex tableaux for basis B is

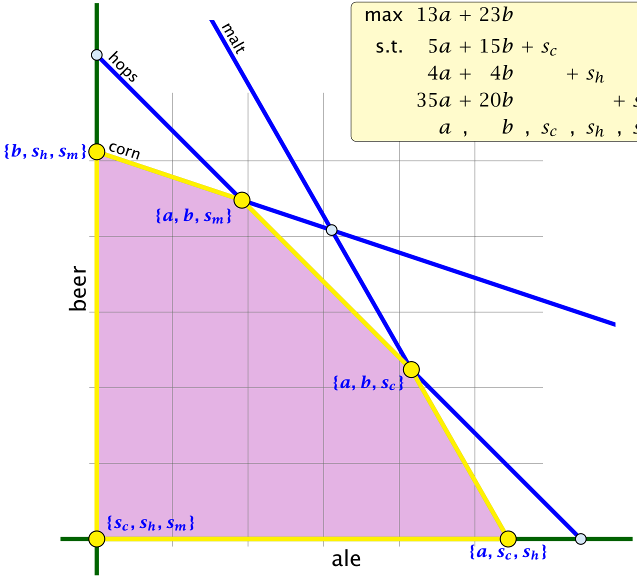
$$\begin{array}{ccccc} & & (c_N^T - c_B^T A_B^{-1} A_N) x_N & = & Z - c_B^T A_B^{-1} b \\ I x_B & + & A_B^{-1} A_N x_N & = & A_B^{-1} b \\ x_B & , & x_N & \geq & 0 \end{array}$$

The BFS is given by $x_N = 0, x_B = A_B^{-1} b$.

If $(c_N^T - c_B^T A_B^{-1} A_N) \leq 0$ we know that we have an optimum solution.

Geometric View of Pivoting

$$
\begin{array}{lrcrcrcrcrcrcl}
\max & 13a & + & 23b & & & & & & & & \\
\text{s.t.} & 5a & + & 15b & + & s_c & & & & & = & 480 \\
 & 4a & + & 4b & & & + & s_h & & & = & 160 \\
 & 35a & + & 20b & & & & & + & s_m & = & 1190 \\
 & a & , & b & , & s_c & , & s_h & , & s_m & \geq & 0
\end{array}
$$



Algebraic Definition of Pivoting

- Given basis B with BFS x^*.
- Choose index $j \notin B$ in order to increase x_j^* from 0 to $\theta > 0$.
  - Other non-basis variables should stay at 0.
  - Basis variables change to maintain feasibility.
- Go from x^* to $x^* + \theta \cdot d$.

Requirements for d:

- $d_j = 1$ (normalization)
- $d_\ell = 0$, $\ell \notin B$, $\ell \neq j$
- $A(x^* + \theta d) = b$ must hold. Hence $Ad = 0$.
- Altogether: $A_B d_B + A_{*j} = Ad = 0$, which gives $d_B = -A_B^{-1} A_{*j}$.

Algebraic Definition of Pivoting

Definition 2 (j-th basis direction)

Let B be a basis, and let $j \notin B$. The vector d with $d_j = 1$ and $d_\ell = 0, \ell \notin B, \ell \neq j$ and $d_B = -A_B^{-1} A_{*j}$ is called the j-th basis direction for B.

Going from x^* to $x^* + \theta \cdot d$ the objective function changes by

$$\theta \cdot c^T d = \theta(c_j - c_B^T A_B^{-1} A_{*j})$$

Algebraic Definition of Pivoting

Definition 3 (Reduced Cost)

For a basis B the value

$$\tilde{c}_j = c_j - c_B^T A_B^{-1} A_{*j}$$

is called the reduced cost for variable x_j.

Note that this is defined for every j. If $j \in B$ then the above term is 0.

Algebraic Definition of Pivoting

Let our linear program be

$$\begin{array}{rcrcl} c_B^T x_B & + & c_N^T x_N & = & Z \\ A_B x_B & + & A_N x_N & = & b \\ x_B & , & x_N & \geq & 0 \end{array}$$

The simplex tableaux for basis B is

$$\begin{array}{rcrcl} & & (c_N^T - c_B^T A_B^{-1} A_N) x_N & = & Z - c_B^T A_B^{-1} b \\ I x_B & + & A_B^{-1} A_N x_N & = & A_B^{-1} b \\ x_B & , & x_N & \geq & 0 \end{array}$$

The BFS is given by $x_N = 0, x_B = A_B^{-1} b$.

If $(c_N^T - c_B^T A_B^{-1} A_N) \leq 0$ we know that we have an optimum solution.

4 Simplex Algorithm

Questions:

- What happens if the min ratio test fails to give us a value θ by which we can safely increase the entering variable?
- How do we find the initial basic feasible solution?
- Is there always a basis B such that

$$(c_N^T - c_B^T A_B^{-1} A_N) \leq 0 \ \ ?$$

  Then we can terminate because we know that the solution is optimal.
- If yes how do we make sure that we reach such a basis?

Min Ratio Test

The min ratio test computes a value $\theta \geq 0$ such that after setting the entering variable to θ the leaving variable becomes 0 and all other variables stay non-negative.

For this, one computes b_i/A_{ie} for all constraints i and calculates the minimum positive value.

What does it mean that the ratio b_i/A_{ie} (and hence A_{ie}) is negative for a constraint?

This means that the corresponding basic variable will increase if we increase b. Hence, there is no danger of this basic variable becoming negative

What happens if **all** b_i/A_{ie} are negative? Then we do not have a leaving variable. Then the LP is unbounded!

Termination

The objective function does not decrease during one iteration of the simplex-algorithm.

Does it always increase?

Termination

The objective function may not increase!

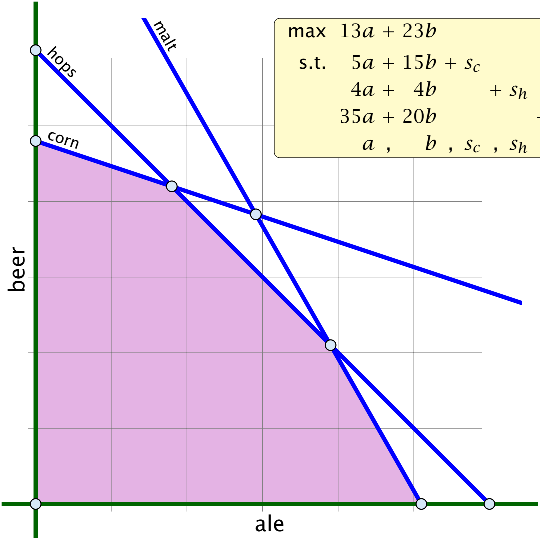
Because a variable x_ℓ with $\ell \in B$ is already 0.

The set of inequalities is degenerate (also the basis is degenerate).

Definition 4 (Degeneracy)
A BFS x^* is called degenerate if the set $J = \{j \mid x_j^* > 0\}$ fulfills $|J| < m$.

It is possible that the algorithm cycles, i.e., it cycles through a sequence of different bases without ever terminating. Happens, very rarely in practise.

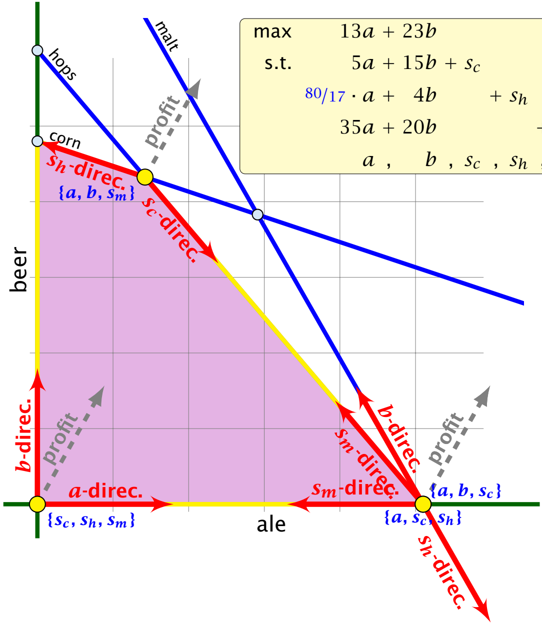
Non Degenerate Example



$$
\begin{array}{llllllll}
\text{max} & 13a & + 23b & & & & & \\
\text{s.t.} & 5a & + 15b & + s_c & & & = & 480 \\
 & 4a & + \ \ 4b & & + s_h & & = & 160 \\
 & 35a & + 20b & & & + s_m & = & 1190 \\
 & a & , \ \ b & , \ s_c & , \ s_h & , \ s_m & \geq & 0
\end{array}
$$

Degenerate Example

$$
\begin{array}{lrl}
\text{max} & 13a + 23b & \\
\text{s.t.} & 5a + 15b + s_c & = 480 \\
 & {}^{80}/_{17} \cdot a + 4b + s_h & = 160 \\
 & 35a + 20b + s_m & = 1190 \\
 & a \ ,\ b \ ,\ s_c \ ,\ s_h \ ,\ s_m & \geq 0
\end{array}
$$



Summary: How to choose pivot-elements

- We can choose a column e as an entering variable if $\tilde{c}_e > 0$ ($\tilde{c}_e$ is reduced cost for x_e).
- The standard choice is the column that maximizes $\tilde{c}_e$.
- If $A_{ie} \leq 0$ for all $i \in \{1, \ldots, m\}$ then the maximum is not bounded.
- Otw. choose a leaving variable ℓ such that $b_\ell / A_{\ell e}$ is minimal among all variables i with $A_{ie} > 0$.
- If several variables have minimum $b_\ell / A_{\ell e}$ you reach a degenerate basis.
- Depending on the choice of ℓ it may happen that the algorithm runs into a cycle where it does not escape from a degenerate vertex.

Termination

What do we have so far?

Suppose we are given an initial feasible solution to an LP. If the LP is non-degenerate then Simplex will terminate.

Note that we either terminate because the min-ratio test fails and we can conclude that the LP is unbounded, or we terminate because the vector of reduced cost is non-positive. In the latter case we have an optimum solution.

How do we come up with an initial solution?

- $Ax \leq b, x \geq 0$, and $\boldsymbol{b} \geq \mathbf{0}$.
- The standard slack from for this problem is $Ax + Is = b, x \geq 0, s \geq 0$, where s denotes the vector of slack variables.
- Then $s = b$, $x = 0$ is a basic feasible solution (how?).
- We directly can start the simplex algorithm.

How do we find an initial basic feasible solution for an arbitrary problem?

Two phase algorithm

Suppose we want to maximize $c^T x$ s.t. $Ax = b, x \geq 0$.

1. Multiply all rows with $b_i < 0$ by -1.
2. maximize $-\sum_i v_i$ s.t. $Ax + Iv = b$, $x \geq 0$, $v \geq 0$ using Simplex. $x = 0$, $v = b$ is initial feasible.
3. If $\sum_i v_i > 0$ then the original problem is infeasible.
4. Otw. you have $x \geq 0$ with $Ax = b$.
5. From this you can get basic feasible solution.
6. Now you can start the Simplex for the original problem.

Optimality

Lemma 5
Let B be a basis and x^* a BFS corresponding to basis B. $\tilde{c} \leq 0$ implies that x^* is an optimum solution to the LP.