Flows

Definition 2
An (s,t)-flow in a (complete) directed graph $G = (V, V \times V, c)$ is a function $f : V \times V \mapsto \mathbb{R}_0^+$ that satisfies

1. For each edge (x,y)
$$0 \le f_{xy} \le c_{xy} \ .$$
(capacity constraints)
2. For each $v \in V \setminus \{s,t\}$
$$\sum_x f_{vx} = \sum_x f_{xv} \ .$$
(flow conservation constraints)

Flows

Definition 3
The value of an (s,t)-flow f is defined as
$$\mathrm{val}(f) = \sum_x f_{sx} - \sum_x f_{xs} \ .$$

Maximum Flow Problem:
Find an (s,t)-flow with maximum value.

LP-Formulation of Maxflow

$$\begin{array}{llrcll}
\max & & \sum_z f_{sz} - \sum_z f_{zs} & & & \\
\text{s.t.} & \forall (z,w) \in V \times V & f_{zw} & \le & c_{zw} & \ell_{zw} \\
 & \forall w \ne s,t & \sum_z f_{zw} - \sum_z f_{wz} & = & 0 & p_w \\
 & & f_{zw} & \ge & 0 &
\end{array}$$

$$\begin{array}{lllcr}
\min & & \sum_{(xy)} c_{xy} \ell_{xy} & & \\
\text{s.t.} & f_{xy}\ (x,y \ne s,t): & 1\ell_{xy} - 1p_x + 1p_y & \ge & 0 \\
 & f_{sy}\ (y \ne s,t): & 1\ell_{sy} + 1p_y & \ge & 1 \\
 & f_{xs}\ (x \ne s,t): & 1\ell_{xs} - 1p_x & \ge & -1 \\
 & f_{ty}\ (y \ne s,t): & 1\ell_{ty} + 1p_y & \ge & 0 \\
 & f_{xt}\ (x \ne s,t): & 1\ell_{xt} - 1p_x & \ge & 0 \\
 & f_{st}: & 1\ell_{st} & \ge & 1 \\
 & f_{ts}: & 1\ell_{ts} & \ge & -1 \\
 & & \ell_{xy} & \ge & 0
\end{array}$$

LP-Formulation of Maxflow

$$\begin{array}{lllcr}
\min & & \sum_{(xy)} c_{xy} \ell_{xy} & & \\
\text{s.t.} & f_{xy}\ (x,y \ne s,t): & 1\ell_{xy} - 1p_x + 1p_y & \ge & 0 \\
 & f_{sy}\ (y \ne s,t): & 1\ell_{sy} - 1 + 1p_y & \ge & 0 \\
 & f_{xs}\ (x \ne s,t): & 1\ell_{xs} - 1p_x + 1 & \ge & 0 \\
 & f_{ty}\ (y \ne s,t): & 1\ell_{ty} - 0 + 1p_y & \ge & 0 \\
 & f_{xt}\ (x \ne s,t): & 1\ell_{xt} - 1p_x + 0 & \ge & 0 \\
 & f_{st}: & 1\ell_{st} - 1 + 0 & \ge & 0 \\
 & f_{ts}: & 1\ell_{ts} - 0 + 1 & \ge & 0 \\
 & & \ell_{xy} & \ge & 0
\end{array}$$

LP-Formulation of Maxflow

$$
\begin{array}{llrcl}
\min & & \sum_{(xy)} c_{xy}\ell_{xy} & & \\
\text{s.t.} & f_{xy}\ (x,y \neq s,t): & 1\ell_{xy} - 1p_x + 1p_y & \geq & 0 \\
& f_{sy}\ (y \neq s,t): & 1\ell_{sy} - p_s + 1p_y & \geq & 0 \\
& f_{xs}\ (x \neq s,t): & 1\ell_{xs} - 1p_x + p_s & \geq & 0 \\
& f_{ty}\ (y \neq s,t): & 1\ell_{ty} - p_t + 1p_y & \geq & 0 \\
& f_{xt}\ (x \neq s,t): & 1\ell_{xt} - 1p_x + p_t & \geq & 0 \\
& f_{st}: & 1\ell_{st} - p_s + p_t & \geq & 0 \\
& f_{ts}: & 1\ell_{ts} - p_t + p_s & \geq & 0 \\
& & \ell_{xy} & \geq & 0
\end{array}
$$

with $p_t = 0$ and $p_s = 1$.

LP-Formulation of Maxflow

$$
\begin{array}{llrcl}
\min & & \sum_{(xy)} c_{xy}\ell_{xy} & & \\
\text{s.t.} & f_{xy}: & 1\ell_{xy} - 1p_x + 1p_y & \geq & 0 \\
& & \ell_{xy} & \geq & 0 \\
& & p_s & = & 1 \\
& & p_t & = & 0
\end{array}
$$

We can interpret the ℓ_{xy} value as assigning a length to every edge.

The value p_x for a variable, then can be seen as the distance of x to t (where the distance from s to t is required to be 1 since $p_s = 1$).

The constraint $p_x \leq \ell_{xy} + p_y$ then simply follows from triangle inequality ($d(x,t) \leq d(x,y) + d(y,t) \Rightarrow d(x,t) \leq \ell_{xy} + d(y,t)$).

One can show that there is an optimum LP-solution for the dual problem that gives an integral assignment of variables.

This means $p_x = 1$ or $p_x = 0$ for our case. This gives rise to a cut in the graph with vertices having value 1 on one side and the other vertices on the other side. The objective function then evaluates the capacity of this cut.

This shows that the Maxflow/Mincut theorem follows from linear programming duality.