Technique 2: Rounding the Dual Solution.

Relaxation for Set Cover

Primal:

$$\begin{array}{lrl} \min & \sum_{i \in I} w_i x_i & \\ \text{s.t. } \forall u & \sum_{i : u \in S_i} x_i & \geq 1 \\ & x_i & \geq 0 \end{array}$$

Dual:

$$\begin{array}{lrl} \max & \sum_{u \in U} y_u & \\ \text{s.t. } \forall i & \sum_{u : u \in S_i} y_u & \leq w_i \\ & y_u & \geq 0 \end{array}$$

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Rounding Algorithm:

Let I denote the index set of sets for which the dual constraint is tight. This means for all $i \in I$

$$\sum_{u:u \in S_i} y_u = w_i$$

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Lemma 3

The resulting index set is an f-approximation.

Proof:

Every $u \in U$ is covered.

- Suppose there is a u that is not covered.
- This means $\sum_{u:u \in S_j} y_u < w_j$ for all sets S_j that contain u.
- But then y_u could be increased in the dual solution without violating any constraint. This is a contradiction to the fact that the dual solution is optimal.

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&\le \sum_u f_u y_u
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Let I denote the solution obtained by the first rounding algorithm and I' be the solution returned by the second algorithm. Then

$$I \subseteq I' \ .$$

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- This means $x_i \geq \frac{1}{f}$.
- Because of Complementary Slackness Conditions the corresponding constraint in the dual must be tight.
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