Part III

Data Structures

Abstract Data Type

An abstract data type (ADT) is defined by an interface of operations or methods that can be performed and that have a defined behavior.

The data types in this lecture all operate on objects that are represented by a [key, value] pair.

- The key comes from a totally ordered set, and we assume that there is an efficient comparison function.
- The value can be anything; it usually carries satellite information important for the application that uses the ADT.

Dynamic Set Operations

▶ $S.\,\mathrm{search}(k)$: Returns pointer to object x from S with $\mathrm{key}[x] = k$ or null.

▶ $S.\,\mathrm{insert}(x)$: Inserts object x into set S. $\mathrm{key}[x]$ must not currently exist in the data-structure.

▶ $S.\,\mathrm{delete}(x)$: Given pointer to object x from S, delete x from the set.

▶ $S.\,\mathrm{minimum}()$: Return pointer to object with smallest key-value in S.

▶ $S.\,\mathrm{maximum}()$: Return pointer to object with largest key-value in S.

▶ $S.\,\mathrm{successor}(x)$: Return pointer to the next larger element in S or null if x is maximum.

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Dynamic Set Operations

- **$S.\,\mathbf{union}(S')$:** Sets $S := S \cup S'$. The set S' is destroyed.

- $S.\,\mathbf{merge}(S')$: Sets $S := S \cup S'$. Requires $S \cap S' = \emptyset$.

- $S.\,\mathbf{split}(k, S')$:
  $S := \{x \in S \mid \text{key}[x] \le k\}$, $S' := \{x \in S \mid \text{key}[x] > k\}$.

- $S.\,\mathbf{concatenate}(S')$: $S := S \cup S'$.
  Requires $\text{key}[S.\,\text{maximum}()] \le \text{key}[S'.\,\text{minimum}()]$.

- $S.\,\mathbf{decrease\text{-}key}(x, k)$: Replace $\text{key}[x]$ by $k \le \text{key}[x]$.

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Examples of ADTs

Stack:

- ▶ $S.\,\textbf{push}(x)$: Insert an element.
- ▶ $S.\,\textbf{pop}()$: Return the element from S that was inserted most recently; delete it from S.
- ▶ $S.\,\textbf{empty}()$: Tell if S contains any object.

Queue:

- ▶ $S.\,\textbf{enqueue}(x)$: Insert an element.
- ▶ $S.\,\textbf{dequeue}()$: Return the element that is longest in the structure; delete it from S.
- ▶ $S.\,\textbf{empty}()$: Tell if S contains any object.

Priority-Queue:

- ▶ $S.\,\textbf{insert}(x)$: Insert an element.
- ▶ $S.\,\textbf{delete-min}()$: Return the element with lowest key-value; delete it from S.

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7 Dictionary

Dictionary:

- ▶ $S.\,\mathbf{insert}(x)$: Insert an element x.
- ▶ $S.\,\mathbf{delete}(x)$: Delete the element pointed to by x.
- ▶ $S.\,\mathbf{search}(k)$: Return a pointer to an element e with $\mathrm{key}[e] = k$ in S if it exists; otherwise return null.

7.1 Binary Search Trees

An (internal) binary search tree stores the elements in a binary tree. Each tree-node corresponds to an element. All elements in the left sub-tree of a node v have a smaller key-value than $\text{key}[v]$ and elements in the right sub-tree have a larger-key value. We assume that all key-values are different.

(External Search Trees store objects only at leaf-vertices)

Examples:
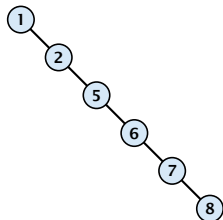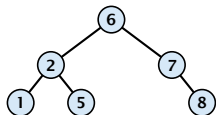
7.1 Binary Search Trees

We consider the following operations on binary search trees. Note that this is a super-set of the dictionary-operations.

- $T.\text{insert}(x)$
- $T.\text{delete}(x)$
- $T.\text{search}(k)$
- $T.\text{successor}(x)$
- $T.\text{predecessor}(x)$
- $T.\text{minimum}()$
- $T.\text{maximum}()$

Binary Search Trees: Searching

Algorithm 1 TreeSearch(x, k)

1: **if** $x = $ null **or** $k = $ key$[x]$ **return** x
2: **if** $k < $ key$[x]$ **return** TreeSearch$($left$[x], k)$
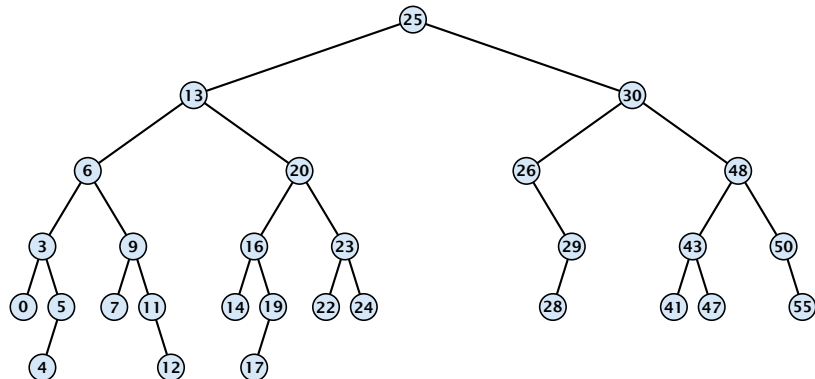3: **else return** TreeSearch$($right$[x], k)$

Binary Search Trees: Searching

TreeSearch(root, 17)

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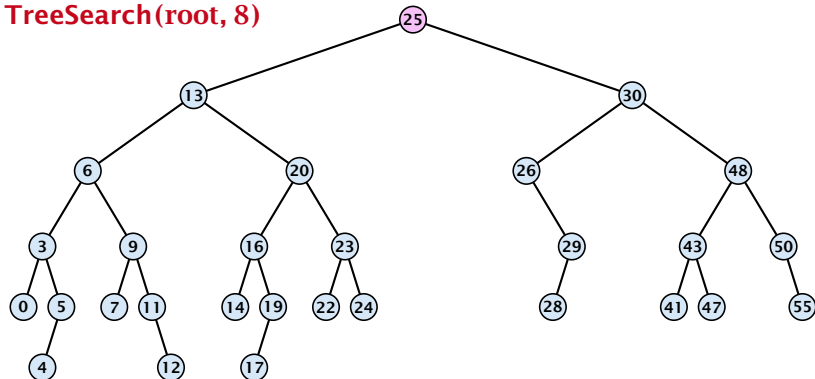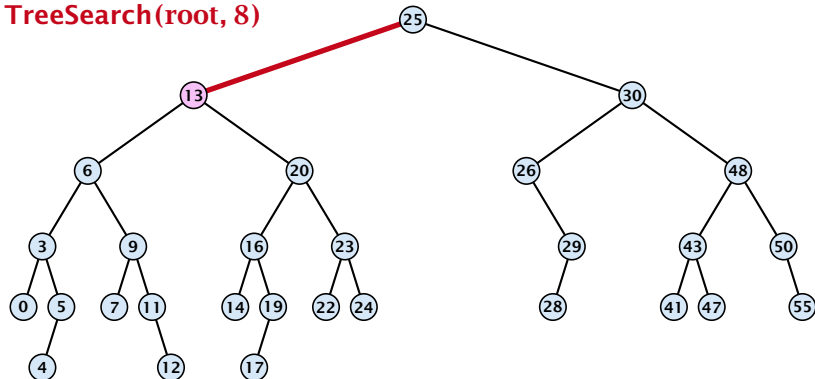

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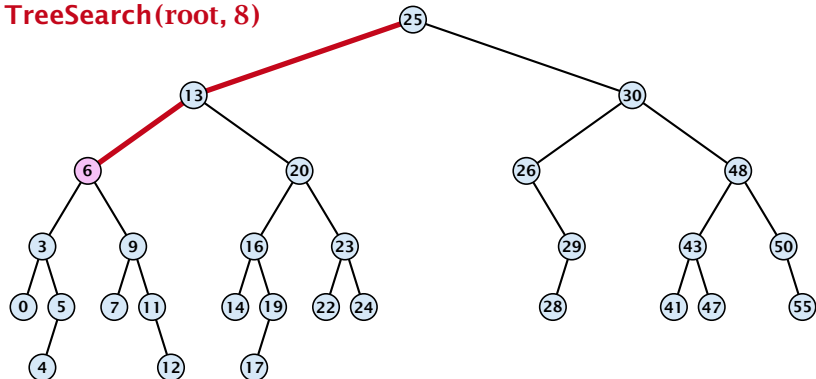**Algorithm 1** TreeSearch(x, k)
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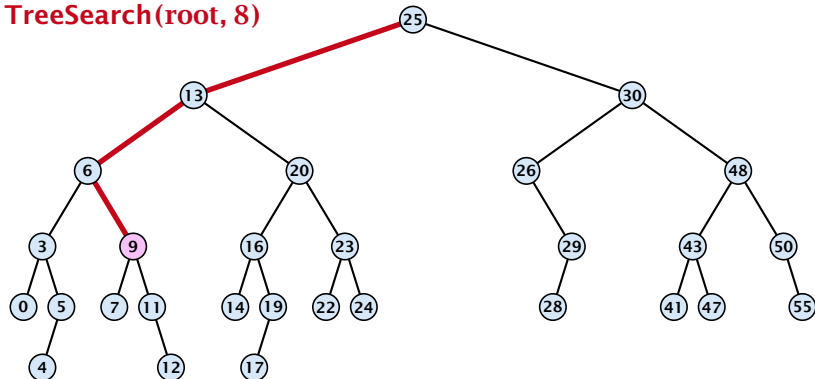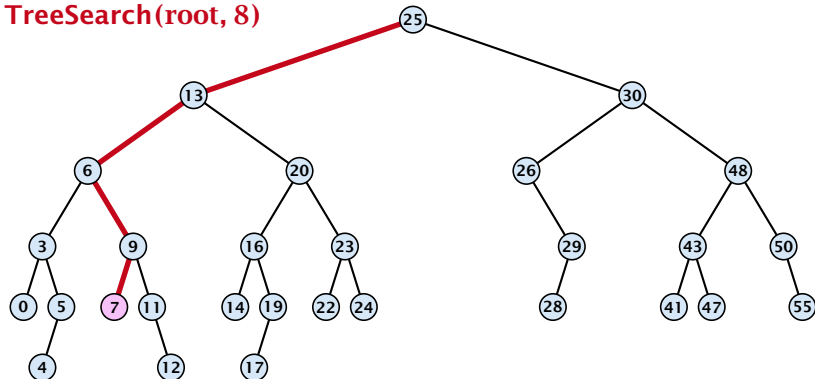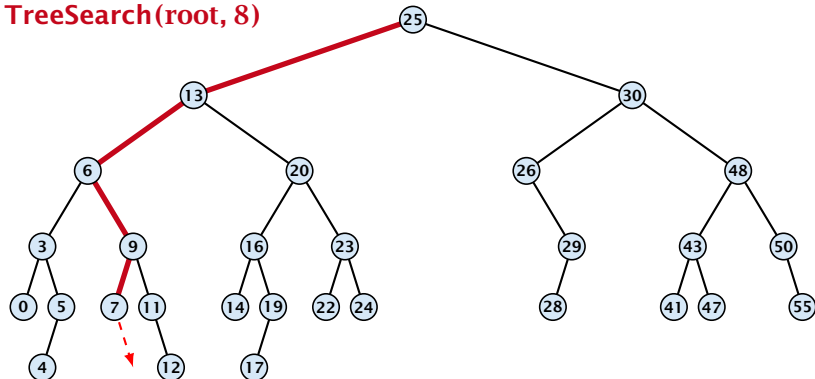3: **else return** TreeSearch(right[x], k)

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Binary Search Trees: Searching

TreeSearch(root, 8)

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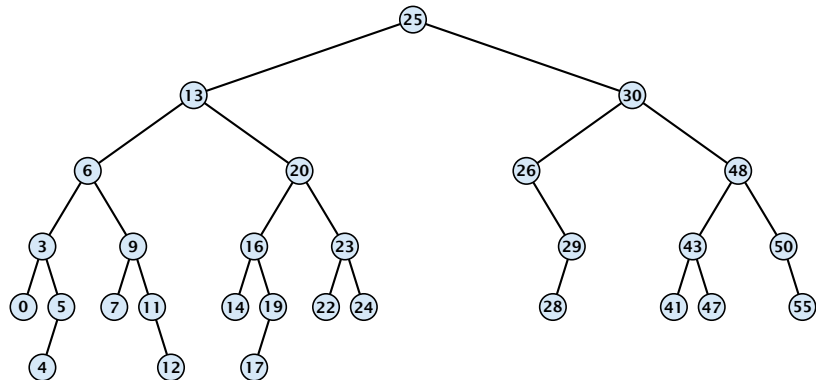3: **else return** TreeSearch(right[x], k)

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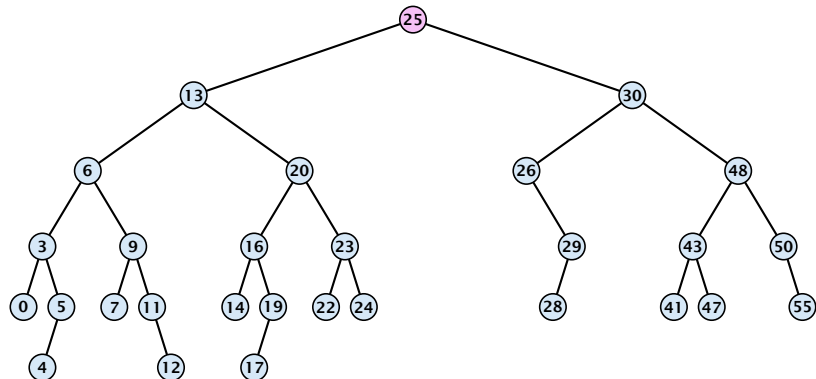3: **else return** TreeSearch(right[x], k)

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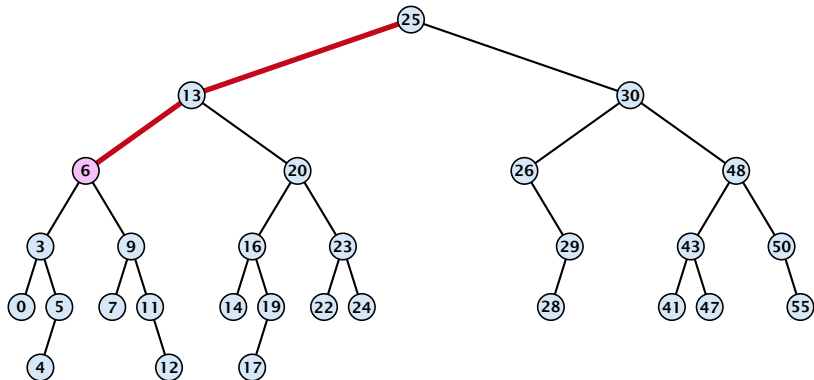2: **if** $k < $ key$[x]$ **return** TreeSearch(left$[x], k$)
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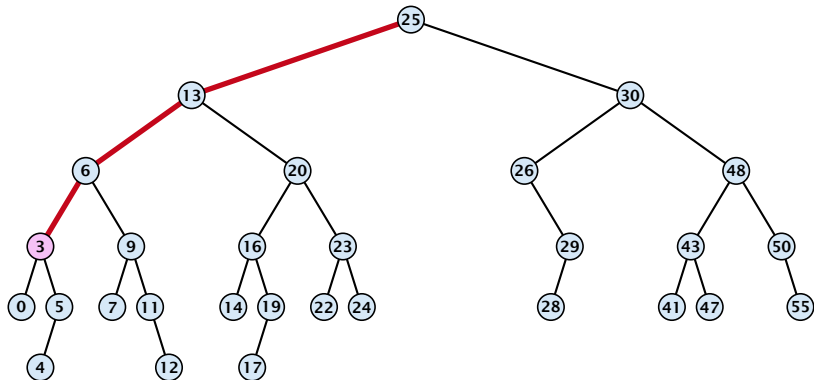
Binary Search Trees: Minimum

Algorithm 2 TreeMin(x)

1: **if** x = null **or** left[x] = null **return** x
2: **return** TreeMin(left[x])

Binary Search Trees: Minimum

Algorithm 2 TreeMin(x)

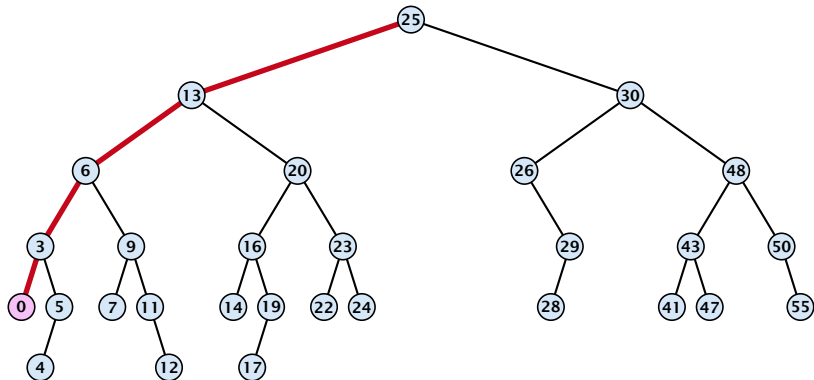1: **if** x = null **or** left[x] = null **return** x
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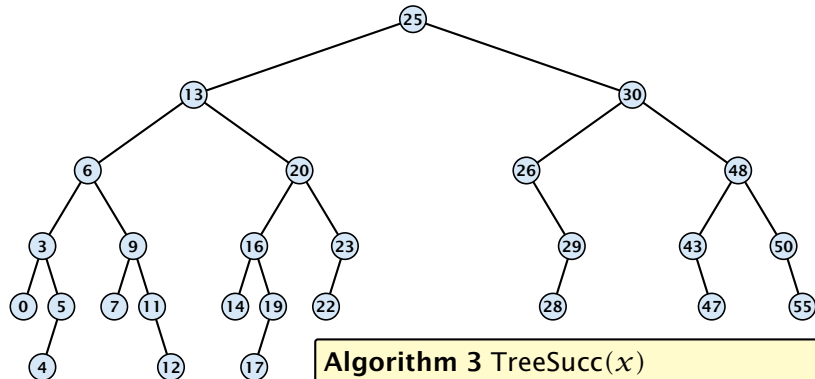2: **return** TreeMin(left[x])

7.1 Binary Search Trees

Binary Search Trees: Minimum

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Binary Search Trees: Successor
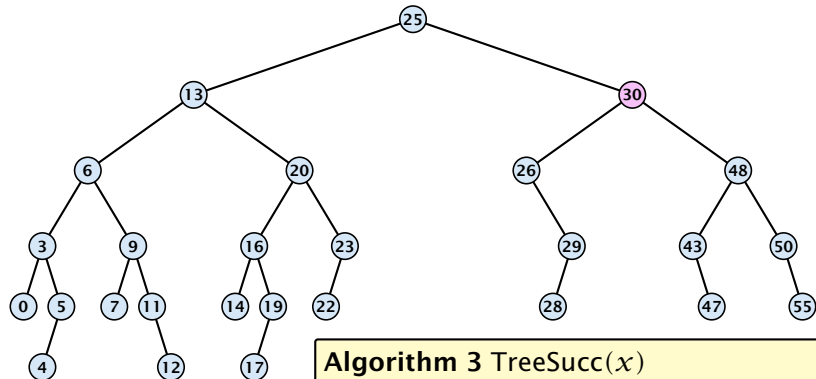

Algorithm 3 TreeSucc(x)

1: **if** right[x] $\neq$ null **return** TreeMin(right[x])
2: $y \leftarrow$ parent[x]
3: **while** $y \neq$ null **and** $x =$ right[y] **do**
4:     $x \leftarrow y$; $y \leftarrow$ parent[x]
5: **return** y;
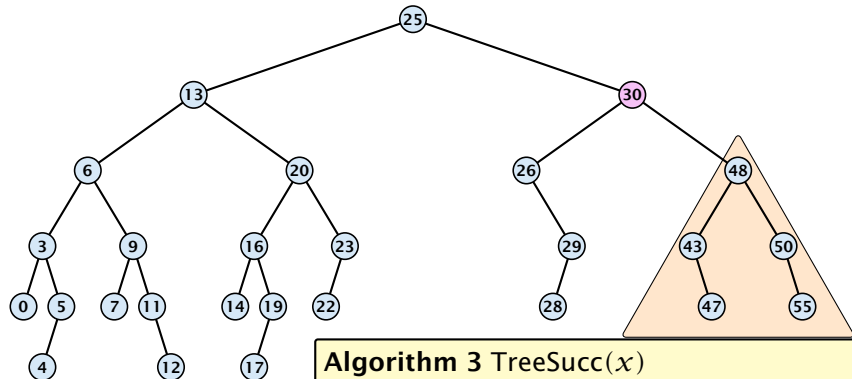
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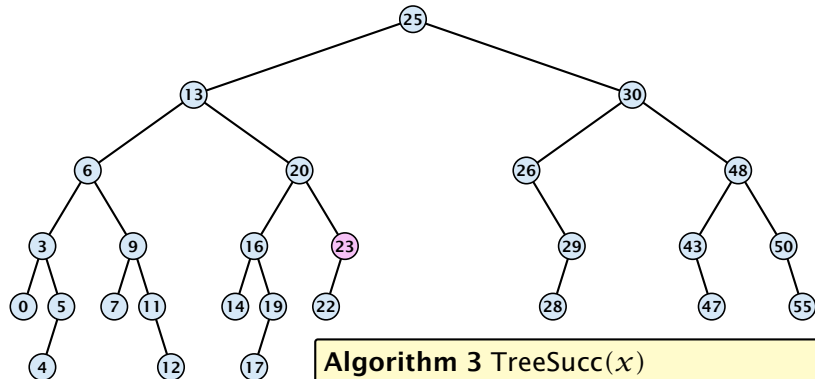4: $x \leftarrow y$; $y \leftarrow$ parent[x]
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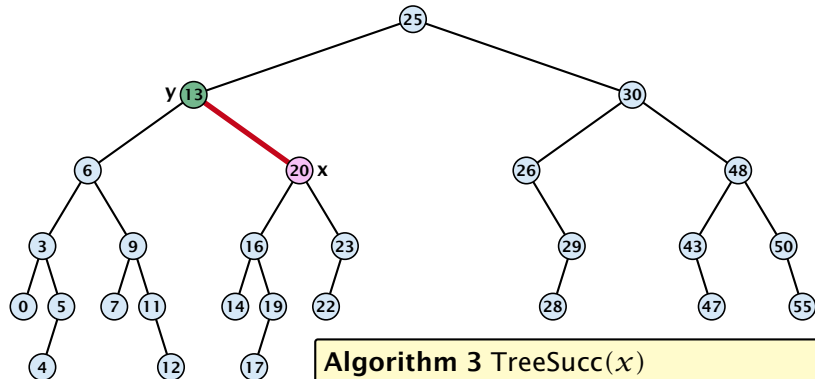4: $\qquad x \leftarrow y; y \leftarrow$ parent[x]
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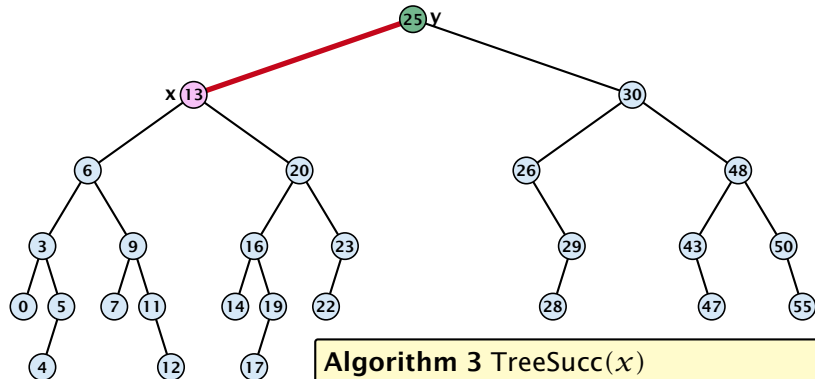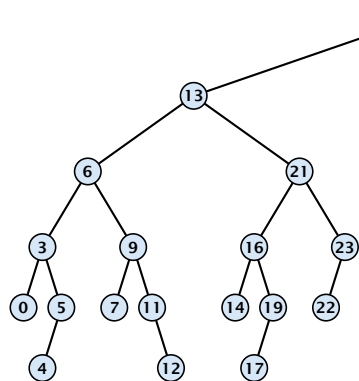4: $x \leftarrow y$; $y \leftarrow$ parent[x]
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Binary Search Trees: Insert

Algorithm 4 TreeInsert(x, z)

1: **if** x = null **then**
2:        root[T] ← z; parent[z] ← null;
3:        **return**;
4: **if** key[x] > key[z] **then**
5:        **if** left[x] = null **then**
6:             left[x] ← z; parent[z] ← x;
7:        **else** TreeInsert(left[x], z);
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Insert element **not** in the tree.
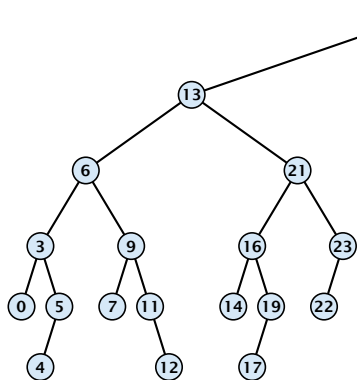
Algorithm 4 TreeInsert(x, z)

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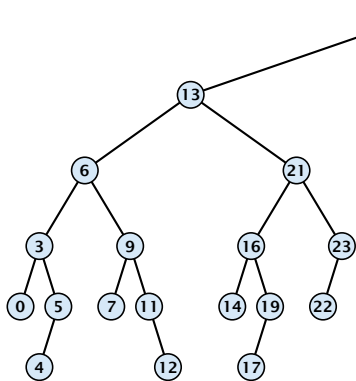Search for z. At some point the search stops at a null-pointer. This is the place to insert z.

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Binary Search Trees: Insert

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TreeInsert(root, 20)

Search for z. At some point the search stops at a null-pointer. This is the place to insert z.

Algorithm 4 TreeInsert(x, z)

1: **if** x = null **then**
2:     root$[T] \leftarrow z$; parent$[z] \leftarrow$ null;
3:     **return**;
4: **if** key$[x] >$ key$[z]$ **then**
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6:         left$[x] \leftarrow z$; parent$[z] \leftarrow x$;
7:     **else** TreeInsert(left$[x], z$);
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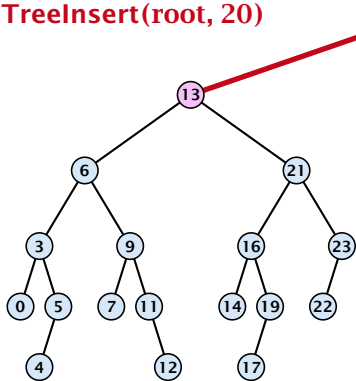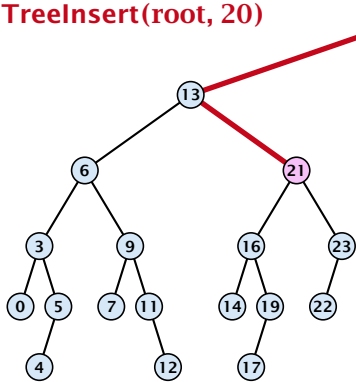Search for z. At some point the search stops at a null-pointer. This is the place to insert z.

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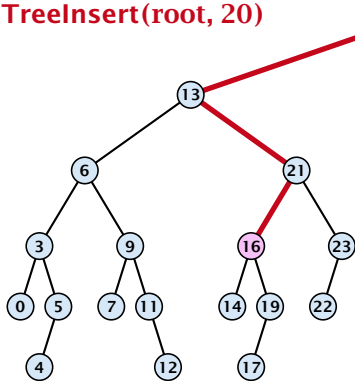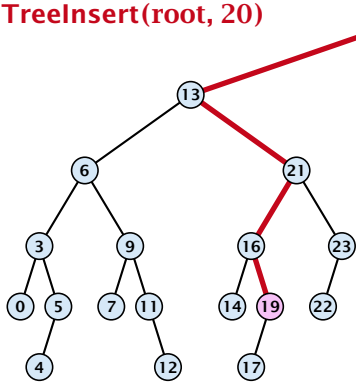at a null-pointer. This
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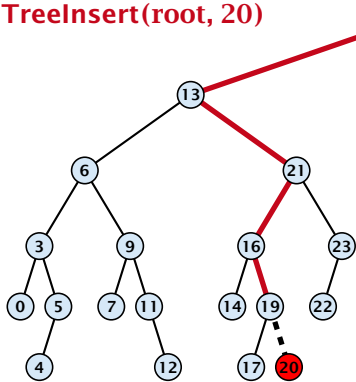Search for z. At some point the search stops at a null-pointer. This is the place to insert z.

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Element does not have any children

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- ▶ Splice the element out of the tree by connecting its parent to its successor.

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Case 3:

Element has two children

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- ▶ Splice successor out of the tree
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Binary Search Trees: Delete

Algorithm 9 TreeDelete(z)

1: **if** left[z] = null **or** right[z] = null
2:      **then** $y \leftarrow z$ **else** $y \leftarrow$ TreeSucc(z);   select y to splice out
3: **if** left[y] $\neq$ null
4:      **then** $x \leftarrow$ left[y] **else** $x \leftarrow$ right[y]; x is child of y (or null)
5: **if** $x \neq$ null **then** parent[x] $\leftarrow$ parent[y];   parent[x] is correct
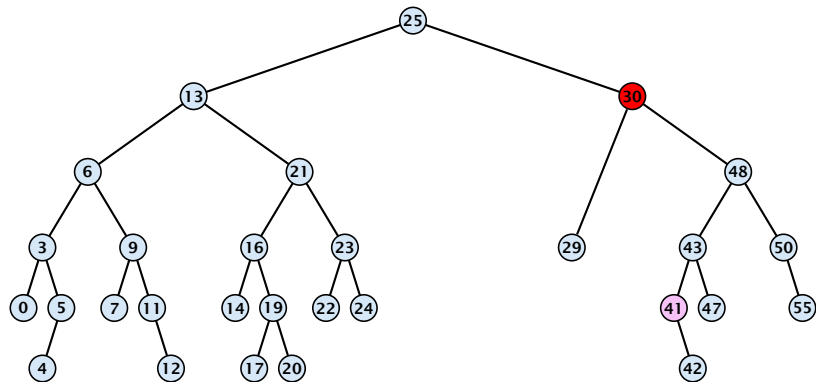6: **if** parent[y] = null **then**
7:      root[T] $\leftarrow x$
8: **else**
9:      **if** y = left[parent[y]] **then**   fix pointer to x
10:         left[parent[y]] $\leftarrow x$
11:      **else**
12:         right[parent[y]] $\leftarrow x$
13: **if** $y \neq z$ **then** copy y-data to z

Balanced Binary Search Trees

All operations on a binary search tree can be performed in time $\mathcal{O}(h)$, where h denotes the height of the tree.

However the height of the tree may become as large as $\Theta(n)$.

Balanced Binary Search Trees
With each insert- and delete-operation perform local adjustments to guarantee a height of $\mathcal{O}(\log n)$.

AVL-trees, Red-black trees, Scapegoat trees, 2-3 trees, B-trees, AA trees, Treaps

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7.2 Red Black Trees

Definition 1

A red black tree is a balanced binary search tree in which each internal node has two children. Each internal node has a color, such that

1. The root is black.
2. All leaf nodes are black.
3. For each node, all paths to descendant leaves contain the same number of black nodes.
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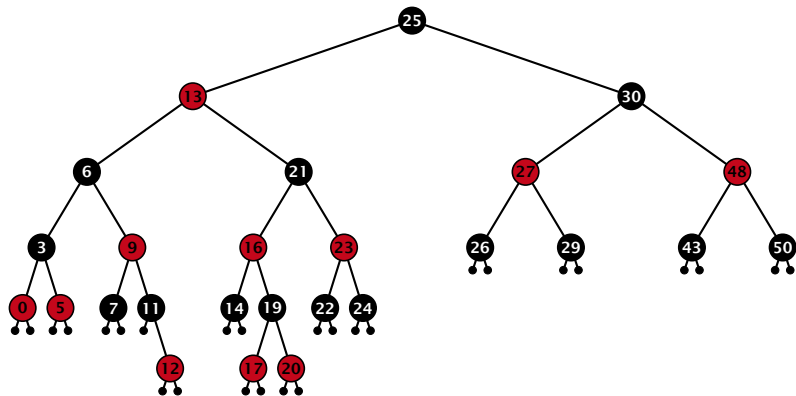
Red Black Trees: Example

7.2 Red Black Trees

Lemma 2

A red-black tree with n internal nodes has height at most $\mathcal{O}(\log n)$.

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The black height $\mathrm{bh}(v)$ of a node v in a red black tree is the number of black nodes on a path from v to a leaf vertex (not counting v).

We first show:

Lemma 4

A sub-tree of black height $\mathrm{bh}(v)$ in a red black tree contains at least $2^{\mathrm{bh}(v)} - 1$ internal vertices.

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Induction on the height of v.

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Proof of Lemma 4.

Induction on the height of ν.

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- ▶ If $\text{height}(\nu)$ (maximum distance btw. ν and a node in the sub-tree rooted at ν) is 0 then ν is a leaf.
- ▶ The black height of ν is 0.
- ▶ The sub-tree rooted at ν contains $0 = 2^{\text{bh}(\nu)} - 1$ inner vertices.

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Induction on the height of v.

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▶ If $\text{height}(v)$ (maximum distance btw. v and a node in the sub-tree rooted at v) is 0 then v is a leaf.

▶ The black height of v is 0.

▶ The sub-tree rooted at v contains $0 = 2^{\text{bh}(v)} - 1$ inner vertices.

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7.2 Red Black Trees

Proof (cont.)

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induction step

- Supose v is a node with $\text{height}(v) > 0$.
- v has two children with strictly smaller height.
- These children (c_1, c_2) either have $\text{bh}(c_i) = \text{bh}(v)$ or $\text{bh}(c_i) = \text{bh}(v) - 1$.
- By induction hypothesis both sub-trees contain at least $2^{\text{bh}(v)-1} - 1$ internal vertices.
- Then T_v contains at least $2(2^{\text{bh}(v)-1} - 1) + 1 \geq 2^{\text{bh}(v)} - 1$ vertices.

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Proof of Lemma 2.

Let h denote the height of the red-black tree, and let P denote a path from the root to the furthest leaf.

At least half of the node on P must be black, since a red node must be followed by a black node.

Hence, the black height of the root is at least $h/2$.

The tree contains at least $2^{h/2} - 1$ internal vertices. Hence, $2^{h/2} - 1 \leq n$.

Hence, $h \leq 2 \log(n + 1) = \mathcal{O}(\log n)$. □

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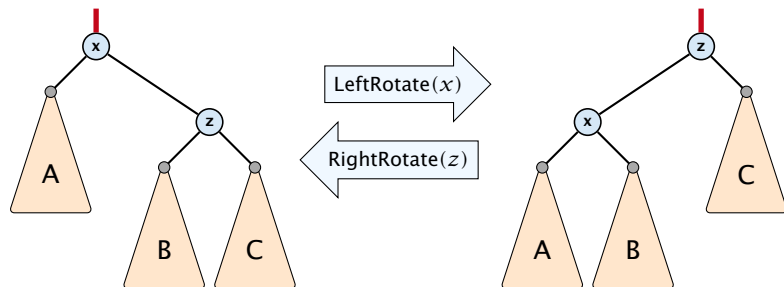
7.2 Red Black Trees

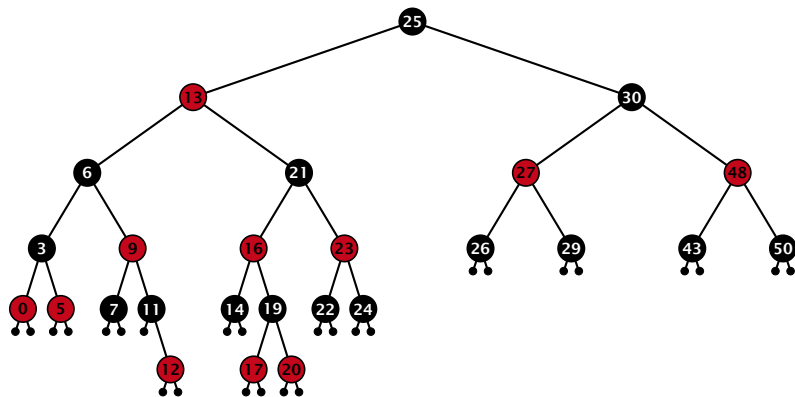We need to adapt the insert and delete operations so that the red black properties are maintained.

Rotations

The properties will be maintained through rotations:
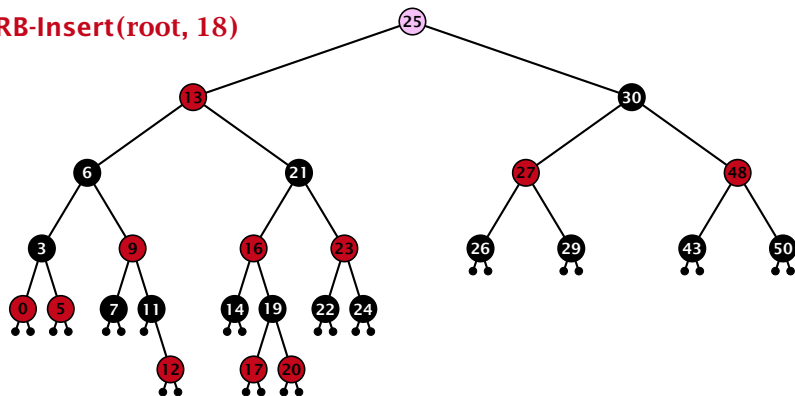
Red Black Trees: Insert

Insert:

- ▶ first make a normal insert into a binary search tree
- ▶ then fix red-black properties

Red Black Trees: Insert

RB-Insert(root, 18)

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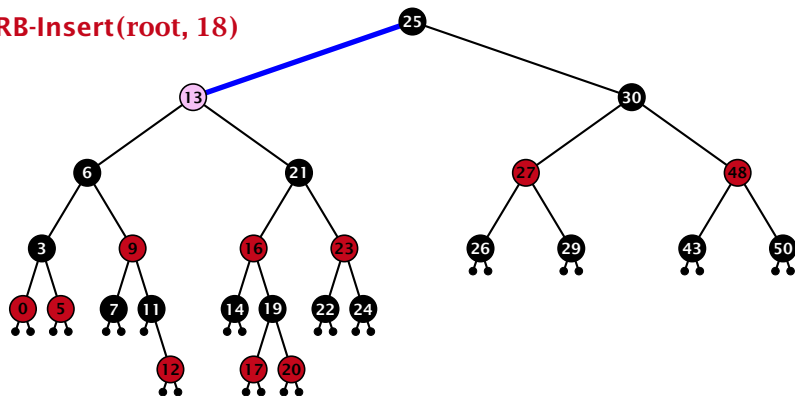
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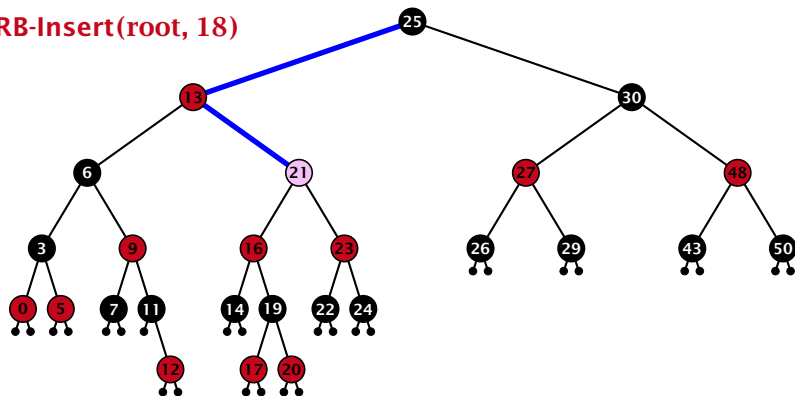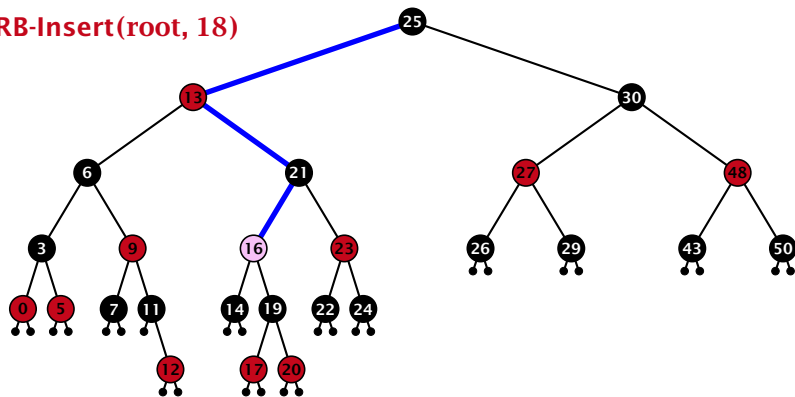

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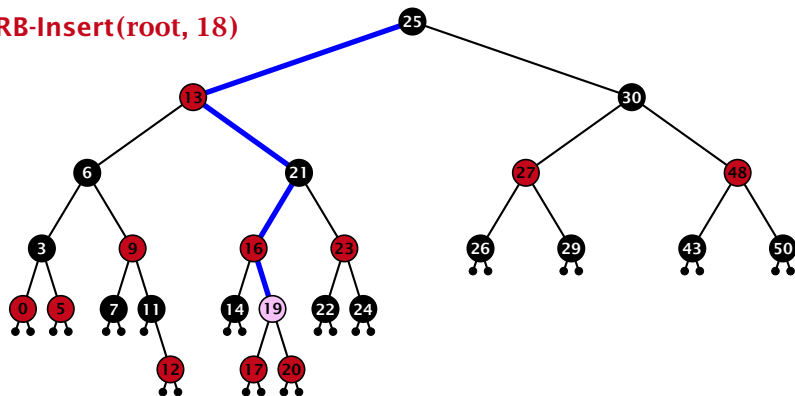▸ then fix red-black properties

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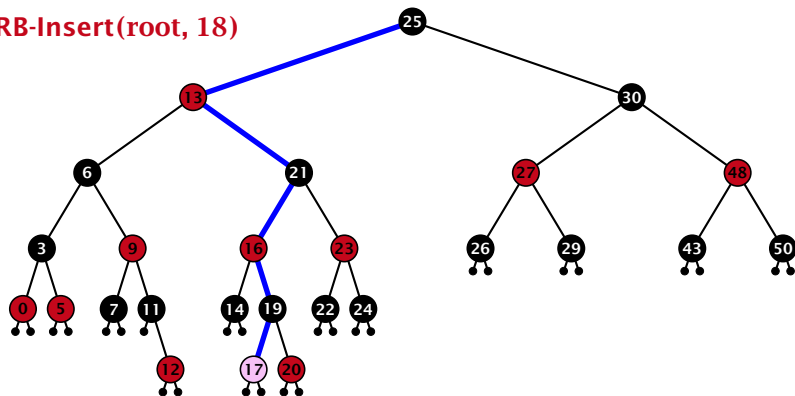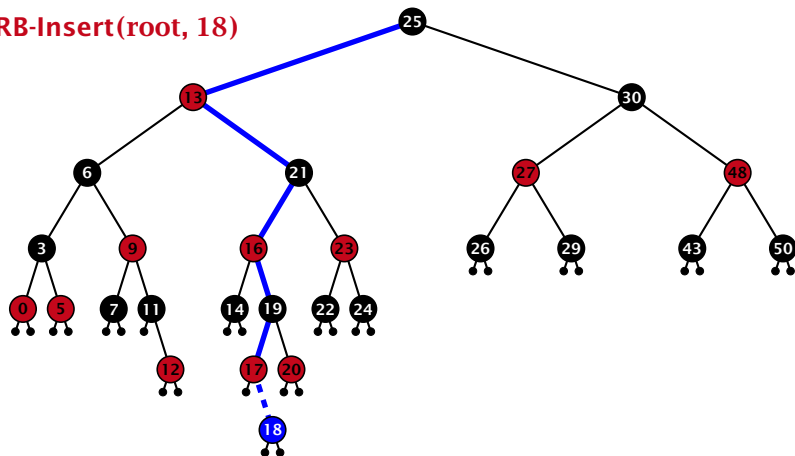**Insert:**

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Red Black Trees: Insert

Invariant of the fix-up algorithm:

- ▶ z is a red node
- ▶ the black-height property is fulfilled at every node
- ▶ the only violation of red-black properties occurs at z and parent[z]

If z has a parent but no grand-parent we could simply color the parent/root black; however this case never happens.

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Algorithm 10 InsertFix(z)

1: **while** parent[z] $\neq$ null **and** col[parent[z]] = red **do**

2:    **if** parent[z] = left[gp[z]] **then**

3:       *uncle* ← right[grandparent[z]]

4:       **if** col[*uncle*] = red **then**

5:          col[p[z]] ← black; col[u] ← black;

6:          col[gp[z]] ← red; z ← grandparent[z];

7:       **else**

8:          **if** z = right[parent[z]] **then**

9:             z ← p[z]; LeftRotate(z);

10:          col[p[z]] ← black; col[gp[z]] ← red;

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13: col(root[T]) ← black;

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1: **while** parent[z] $\neq$ null **and** col[parent[z]] = red **do**
2:     **if** parent[z] = left[gp[z]] **then** *z in left subtree of grandparent*
3:         *uncle* ← right[grandparent[z]]
4:         **if** col[*uncle*] = red **then**
5:             col[p[z]] ← black; col[u] ← black;
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| 9: $z \leftarrow$ p[z]; LeftRotate(z); |
| 10:   col[p[z]] $\leftarrow$ black; col[gp[z]] $\leftarrow$ red; 2b: z left child |
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Case 1: Red Uncle
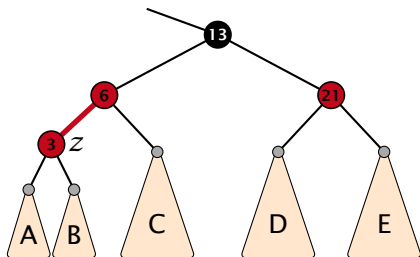
Case 1: Red Uncle
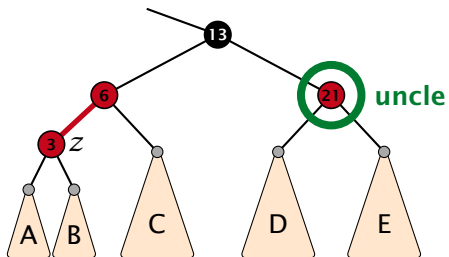
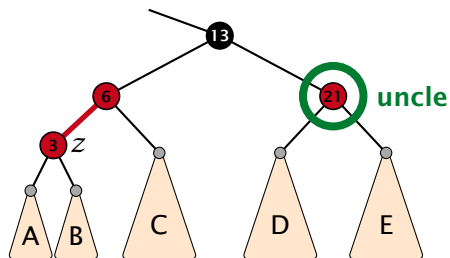# Case 1: Red Uncle
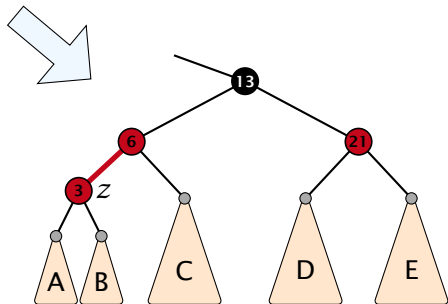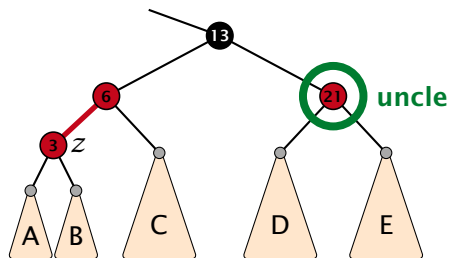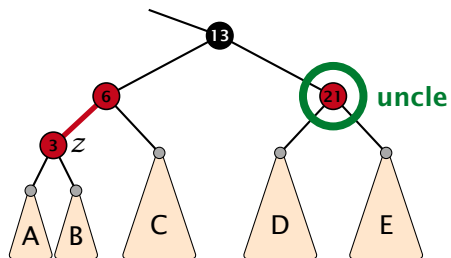
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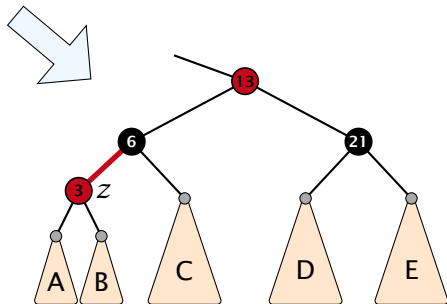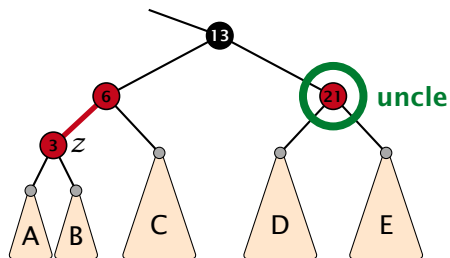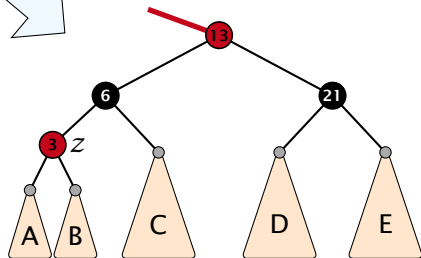
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7.2 Red Black Trees

Case 1: Red Uncle

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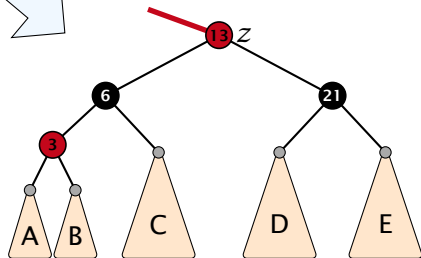2. move z to grand-parent
3. invariant is fulfilled for new z

7.2 Red Black Trees

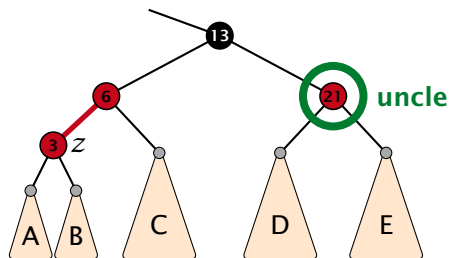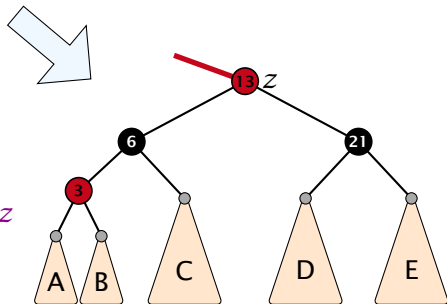# Case 1: Red Uncle
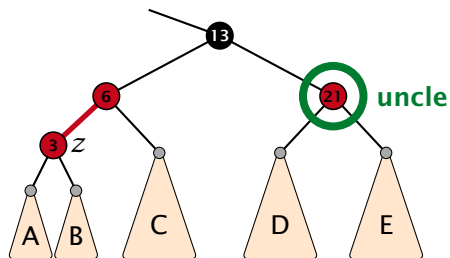
1. recolour
2. move *z* to grand-parent
3. invariant is fulfilled for new *z*
4. you made progress

1. rotate around grandparent
2. re-colour to ensure that black height property holds
3. you have a red black tree

Case 2b: Black uncle and *z* is left child

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Case 2a: Black uncle and z is right child

1. rotate around parent
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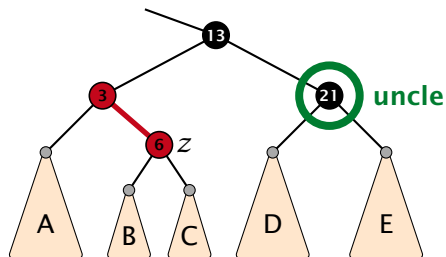
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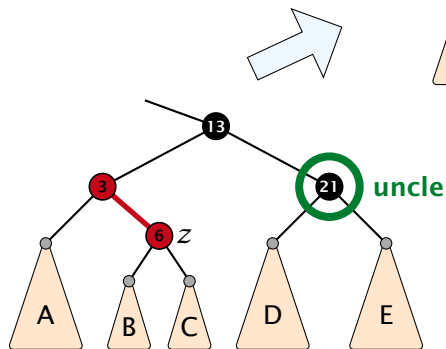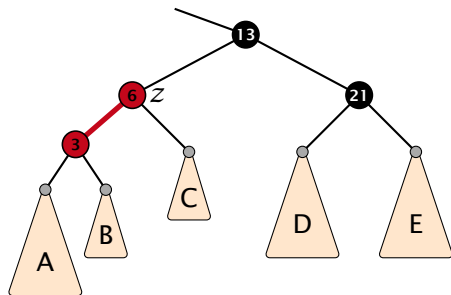
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Running time:

► Only Case 1 may repeat; but only $h/2$ many steps, where h is the height of the tree.

► Case 2a → Case 2b → red-black tree

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Performing Case 1 at most $\mathcal{O}(\log n)$ times and every other case at most once, we get a red-black tree. Hence $\mathcal{O}(\log n)$ re-colorings and at most 2 rotations.

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- If you delete the root, the root may now be red.

- Every path from an ancestor of x to a descendant leaf of x changes the number of black nodes. Black height property might be violated.

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Case 3:

Element has two children

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- ▸ when replacing content by content of successor, don't change color of node

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Invariant of the fix-up algorithm

▶ the node z is black

▶ if we "assign" a fake black unit to the edge from z to its parent then the black-height property is fulfilled

Goal: make rotations in such a way that you at some point can remove the fake black unit from the edge.

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Case 1: Sibling of z is red

1. left-rotate around parent of z

2. recolor nodes b and c

3. the new sibling is black
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4. Case 2 (special),
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Case 2: Sibling is black with two black children

1. re-color node c
2. move fake black unit upwards
3. move z upwards
4. we made progress
5. if b is red we color it black and are done

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Case 3: Sibling black with one black child to the right

1. do a right-rotation at sibling
2. recolor c and d
3. new sibling is black with red right child (Case 4)

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Case 3: Sibling black with one black child to the right

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Case 4: Sibling is black with red right child

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2. remove the fake black unit
3. recolor nodes b, c, and e
4. you have a valid
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Splay Trees

Disadvantage of balanced search trees:

- worst case; no advantage for easy inputs
- additional memory required
- complicated implementation

Splay Trees:

- after accessing an element it is moved to the root; operation is called splay
- repeated accesses are faster
- only amortized guarantee
- read-operations change the tree

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Splay Trees

find(x)

- search for x according to a search tree
- let $\bar{x}$ be last element on search-path
- splay($\bar{x}$)

Splay Trees

insert(x)

- ▶ search for x; $\bar{x}$ is last visited element during search (successer or predecessor of x)
- ▶ splay($\bar{x}$) moves $\bar{x}$ to the root
- ▶ insert x as new root
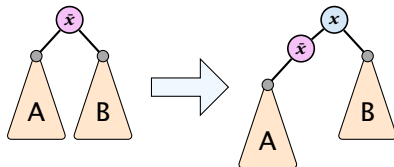
Splay Trees

delete(x)

- ▶ search for x; splay(x); remove x
- ▶ search largest element $\bar{x}$ in A
- ▶ splay($\bar{x}$) (on subtree A)
- ▶ connect root of B as right child of $\bar{x}$
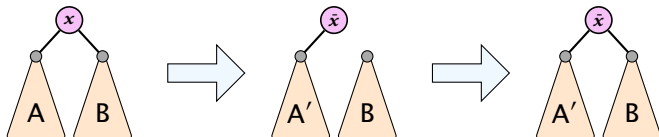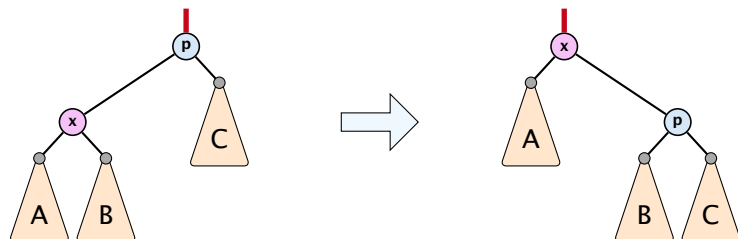
How to bring element to root?

▸ one (bad) option: moveToRoot(x)

▸ iteratively do rotation around parent of x until x is root

▸ if x is left child do right rotation otw. left rotation

Splay: Zig Case
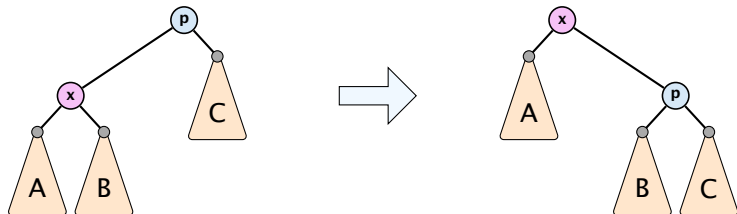
better option splay(x):

▶ zig case: if x is child of root do left rotation or right rotation around parent

Splay: Zigzag Case
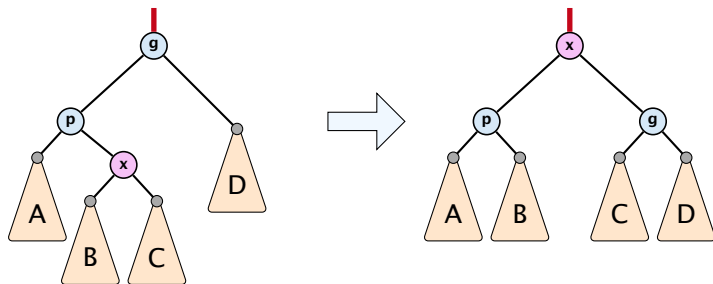
better option splay(x):

- ▶ zigzag case: if x is right child and parent of x is left child (or x left child parent of x right child)
- ▶ do double right rotation around grand-parent (resp. double left rotation)

Double Rotations
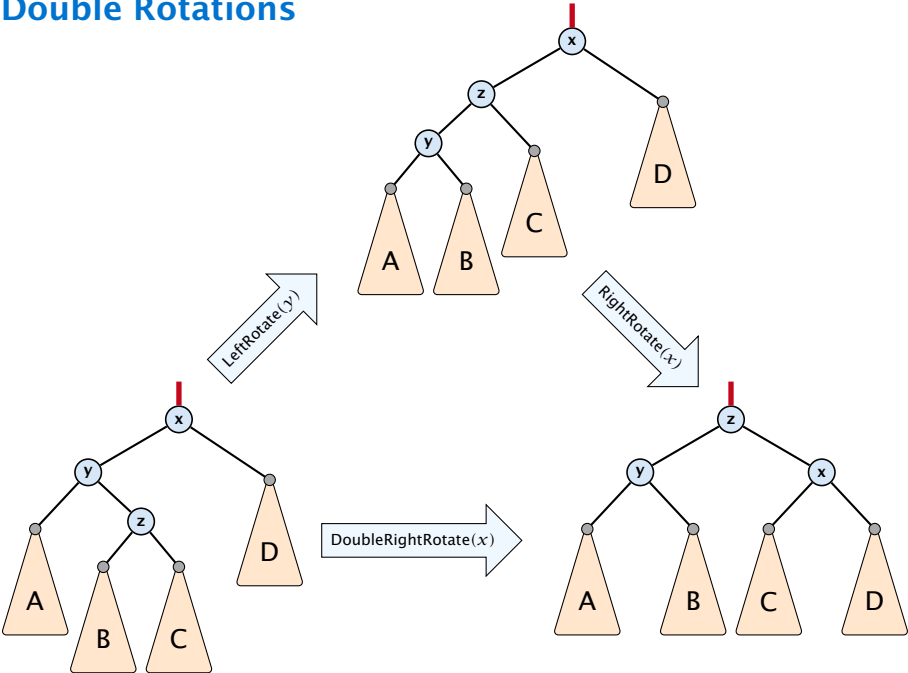
Splay: Zigzig Case
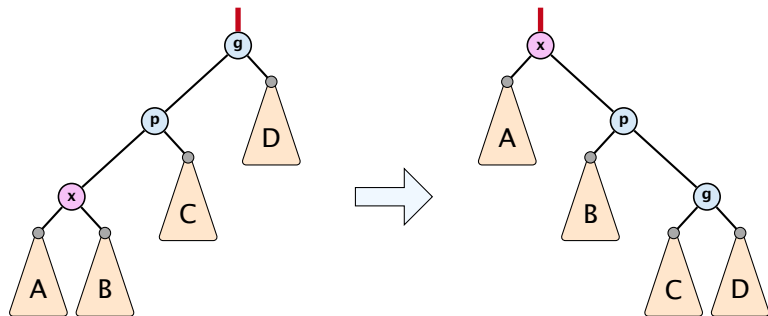
better option splay(x):

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- ▶ do right roation around grand-parent followed by right rotation around parent (resp. left rotations)

Splay vs. Move to Root
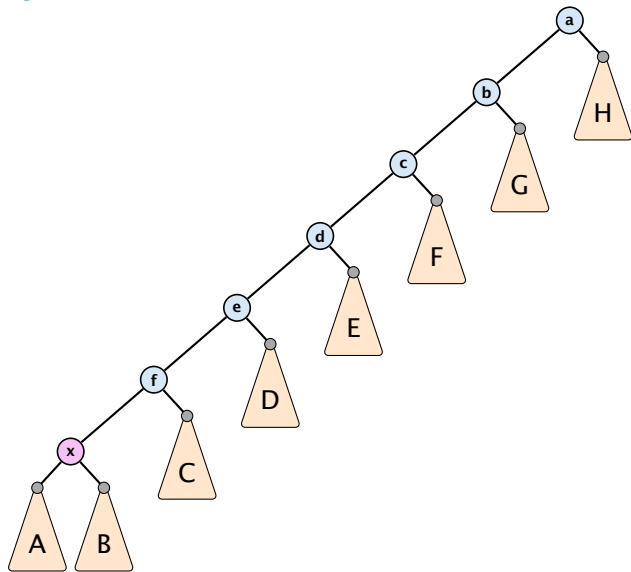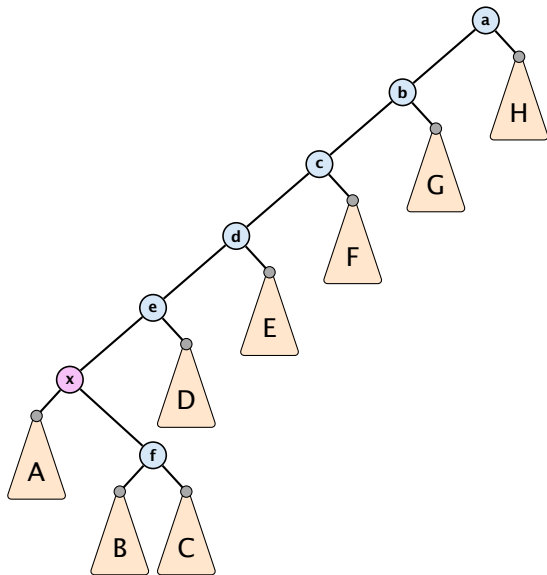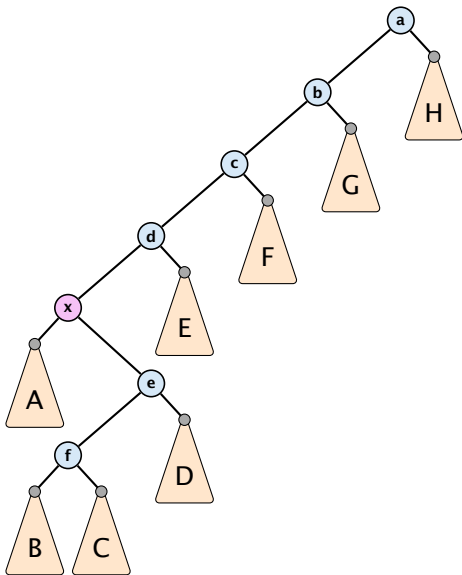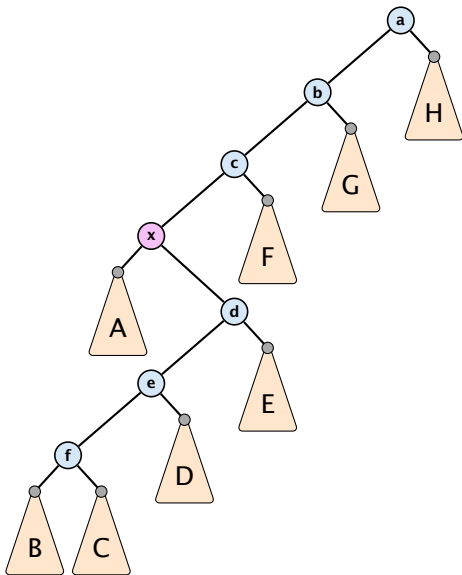
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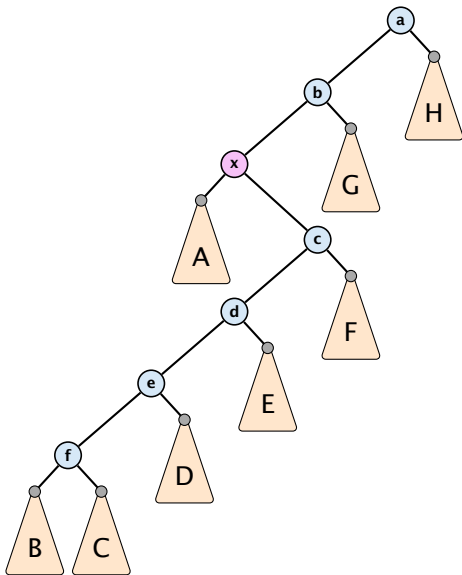
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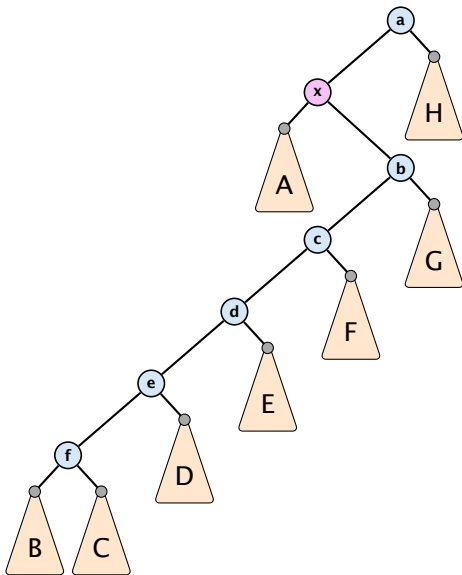
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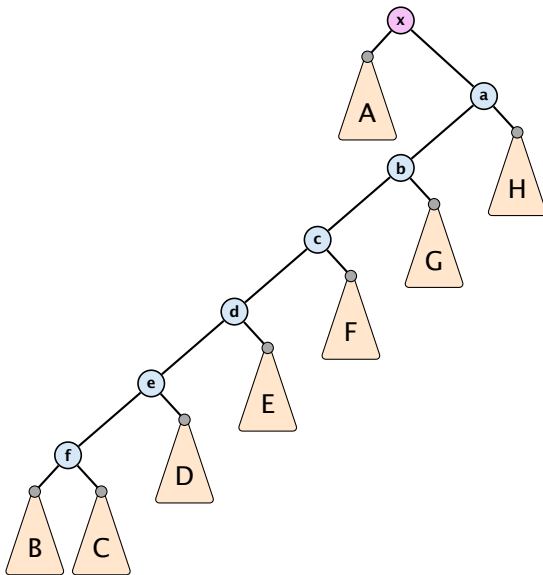
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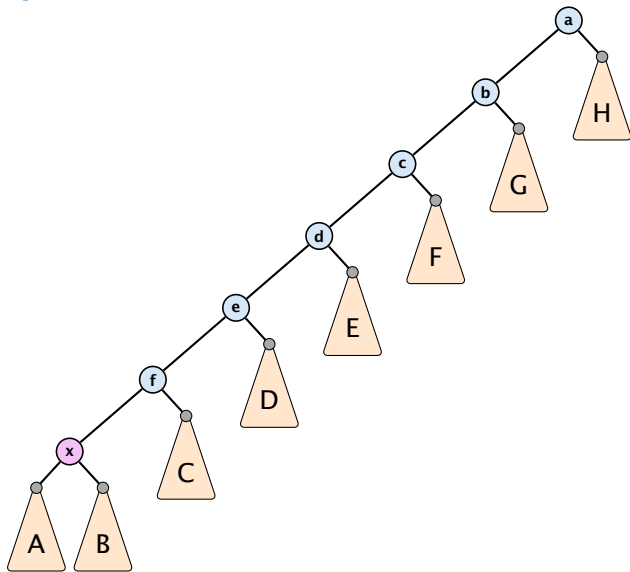
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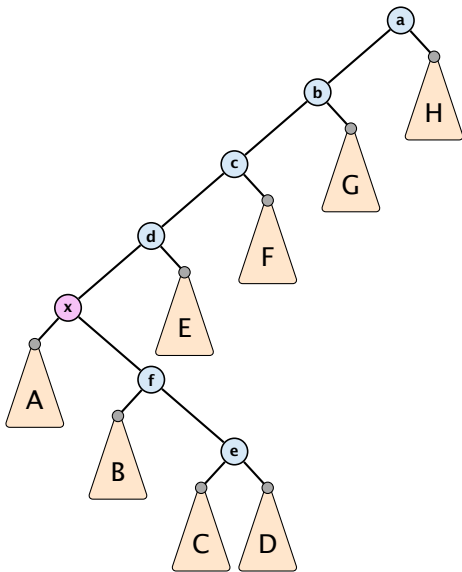
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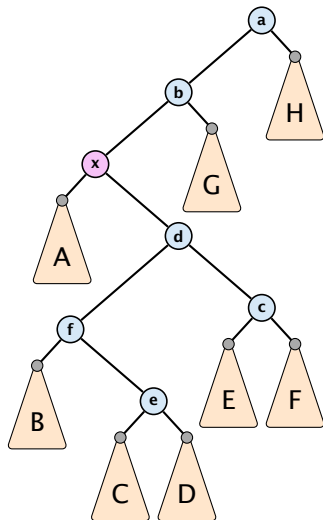
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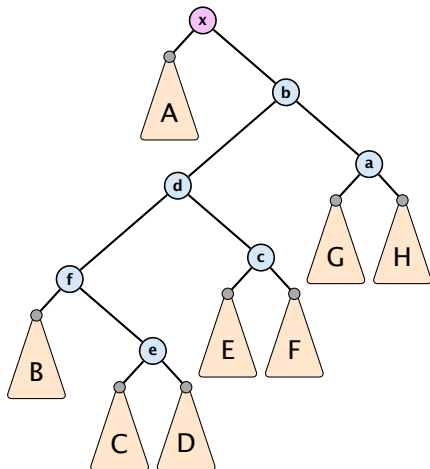
Static Optimality

Suppose we have a sequence of m find-operations. $\text{find}(x)$ appears h_x times in this sequence.

The cost of a static search tree T is:

$$\text{cost}(T) = m + \sum_x h_x \, \text{depth}_T(x)$$

The total cost for processing the sequence on a splay-tree is $\mathcal{O}(\text{cost}(T_{\min}))$, where $T_{\min}$ is an optimal static search tree.

Dynamic Optimality

Let S be a sequence with m find-operations.

Let A be a data-structure based on a search tree:

- ▶ the cost for accessing element x is $1 + \text{depth}(x)$;
- ▶ after accessing x the tree may be re-arranged through rotations;

Conjecture:
A splay tree that only contains elements from S has cost $\mathcal{O}(\text{cost}(A, S))$, for processing S.

Lemma 5

Splay Trees have an amortized running time of $\mathcal{O}(\log n)$ for all operations.

Amortized Analysis

Definition 6

A data structure with operations $\mathrm{op}_1(), \ldots, \mathrm{op}_k()$ has amortized running times $t_1, \ldots, t_k$ for these operations if the following holds.

Suppose you are given a sequence of operations (starting with an empty data-structure) that operate on at most n elements, and let k_i denote the number of occurences of $\mathrm{op}_i()$ within this sequence. Then the actual running time must be at most $\sum_i k_i \cdot t_i(n)$.

Potential Method

Introduce a potential for the data structure.

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Then

$$\sum_{i=1}^{k} c_i \leq \sum_{i=1}^{k} c_i + \Phi(D_k) - \Phi(D_0) = \sum_{i=1}^{k} \hat{c}_i$$

This means the amortized costs can be used to derive a bound on the total cost.

Stack

- ▶ *S*. **push()**
- ▶ *S*. **pop()**
- ▶ *S*. **multipop(k):** removes k items from the stack. If the stack currently contains less than k items it empties the stack.
- ▶ The user has to ensure that pop and multipop do not generate an underflow.

Actual cost:

- ▶ *S*. push(): cost 1.
- ▶ *S*. pop(): cost 1.
- ▶ *S*. multipop(k): cost min{size, k} = k.

Example: Stack

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Example: Binary Counter

Incrementing a binary counter:
Consider a computational model where each bit-operation costs one time-unit.

Incrementing an n-bit binary counter may require to examine n-bits, and maybe change them.

Actual cost:

- Changing bit from 0 to 1: cost 1.
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- Increment: cost is $k + 1$, where k is the number of consecutive ones in the least significant bit-positions (e.g, 001101 has $k = 1$).

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Hence, the amortized cost is $k\hat{C}_{1\to0} + \hat{C}_{0\to1} \le 2$.

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Splay Trees

potential function for splay trees:

- size $s(x) = |T_x|$
- rank $r(x) = \log_2(s(x))$
- $\Phi(T) = \sum_{v \in T} r(v)$

amortized cost = real cost + potential change

The cost is essentially the cost of the splay-operation, which is 1 plus the number of rotations.

Splay: Zig Case

$$\Delta\Phi = r'(x) + r'(p) - r(x) - r(p)$$
$$= r'(p) - r(x)$$
$$\leq r'(x) - r(x)$$

$$\text{cost}_{\text{zig}} \leq 1 + 3(r'(x) - r(x))$$

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Splay: Zigzig Case

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Amortized cost of the whole splay operation:

$$\leq 1 + 1 + \sum_{\text{steps } t} 3(r_t(x) - r_{t-1}(x))$$

$$= 2 + r(\text{root}) - r_0(x)$$

$$\leq \mathcal{O}(\log n)$$

7.4 Augmenting Data Structures

Suppose you want to develop a data structure with:

- ▶ **Insert(x):** insert element x.
- ▶ **Search(k):** search for element with key k.
- ▶ **Delete(x):** delete element referenced by pointer x.
- ▶ **find-by-rank(ℓ):** return the ℓ-th element; return "error" if the data-structure contains less than ℓ elements.

Augment an existing data-structure instead of developing a new one.

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7.4 Augmenting Data Structures

How to augment a data-structure

1. choose an underlying data-structure

2. determine additional information to be stored in the underlying structure

3. verify/show how the additional information can be maintained for the basic modifying operations on the underlying structure.

4. develop the new operations

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Goal: Design a data-structure that supports insert, delete, search, and find-by-rank in time $\mathcal{O}(\log n)$.

1. We choose a red-black tree as the underlying data-structure.

2. We store in each node v the size of the sub-tree rooted at v.

3. We need to be able to update the size-field in each node without asymptotically affecting the running time of insert, delete, and search. We come back to this step later...

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4. How does find-by-rank work?
   Find-by-rank(k) := Select(root, k) with

Algorithm 7 Select(x, i)

1: **if** $x = $ null **then return** error
2: **if** left[x] $\neq$ null **then** $r \leftarrow$ left[x]. size $+1$ **else** $r \leftarrow 1$
3: **if** $i = r$ **then return** x
4: **if** $i < r$ **then**
5:      **return** Select(left[x], i)
6: **else**
7:      **return** Select(right[x], $i - r$)

Select(x, i)

Find-by-rank:

▶ decide whether you have to proceed into the left or right sub-tree

▶ adjust the rank that you are searching for if you go right

Select(x, i)

Select(25, 14)

Find-by-rank:

- decide whether you have to proceed into the left or right sub-tree
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Select(x, i)

Select(13, 14)

Find-by-rank:

▶ decide whether you have to proceed into the left or right sub-tree

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Select(x, i)

Select(21, 5)

Find-by-rank:

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Select(x, i)

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Select(x, i)

Select(19, 3)

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7.4 Augmenting Data Structures

Goal: Design a data-structure that supports insert, delete, search, and find-by-rank in time $\mathcal{O}(\log n)$.

3. How do we maintain information?

Search(k): Nothing to do.

Insert(x): When going down the search path increase the size field for each visited node. Maintain the size field during rotations.

Delete(x): Directly after splicing out a node traverse the path from the spliced out node upwards, and decrease the size counter on every node on this path. Maintain the size field during rotations.

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Rotations

The only operation during the fix-up procedure that alters the tree and requires an update of the size-field:
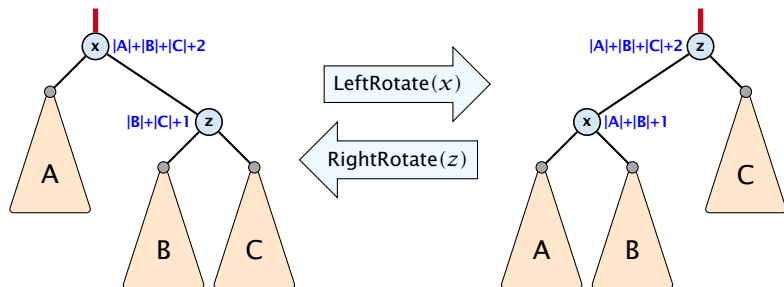
The nodes x and z are the only nodes changing their size-fields.

The new size-fields can be computed locally from the size-fields of the children.

7.5 (a, b)-trees

Definition 7
For $b \geq 2a - 1$ an (a, b)-tree is a search tree with the following properties

1. all leaves have the same distance to the root
2. every internal non-root vertex v has at least a and at most b children
3. the root has degree at least 2 if the tree is non-empty
4. the internal vertices do not contain data, but only keys (external search tree)
5. there is a special dummy leaf node with key-value ∞

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7.5 (a, b)-trees

Each internal node v with $d(v)$ children stores $d - 1$ keys $k_1, \ldots, k_{d-1}$. The i-th subtree of v fulfills

$$k_{i-1} < \text{ key in } i\text{-th sub-tree } \le k_i \ ,$$

where we use $k_0 = -\infty$ and $k_d = \infty$.

7.5 (a, b)-trees

Example 8

7.5 (a, b)-trees

Variants

- ▶ The dummy leaf element may not exist; it only makes implementation more convenient.

- ▶ Variants in which $b = 2a$ are commonly referred to as B-trees.

- ▶ A B-tree usually refers to the variant in which keys and data are stored at internal nodes.

- ▶ A B^+ tree stores the data only at leaf nodes as in our definition. Sometimes the leaf nodes are also connected in a linear list data structure to speed up the computation of successors and predecessors.

- ▶ A B^* tree requires that a node is at least $2/3$-full as opposed to $1/2$-full (the requirement of a B-tree).

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- ▶ A B-tree usually refers to the variant in which keys and data are stored at internal nodes.

- ▶ A B^+ tree stores the data only at leaf nodes as in our definition. Sometimes the leaf nodes are also connected in a linear list data structure to speed up the computation of successors and predecessors.

- ▶ A B^* tree requires that a node is at least $2/3$-full as opposed to $1/2$-full (the requirement of a B-tree).

Lemma 9

Let T be an (a, b)-tree for $n > 0$ elements (i.e., $n + 1$ leaf nodes) and height h (number of edges from root to a leaf vertex). Then

1. $2a^{h-1} \leq n + 1 \leq b^h$

2. $\log_b(n + 1) \leq h \leq 1 + \log_a\left(\frac{n+1}{2}\right)$

Proof.

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Proof.

- If $n > 0$ the root has degree at least 2 and all other nodes have degree at least a. This gives that the number of leaf nodes is at least $2a^{h-1}$.
- Analogously, the degree of any node is at most b and, hence, the number of leaf nodes at most b^h.

□

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Search
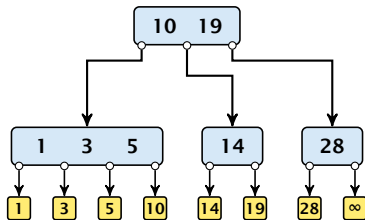
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Search(8)
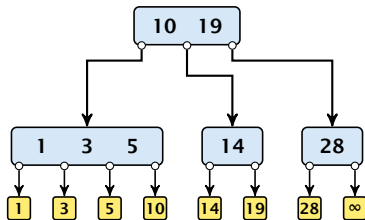
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Search(8)
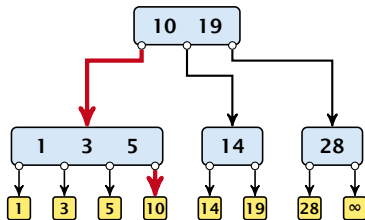
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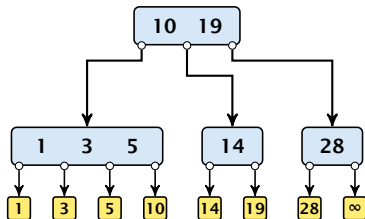
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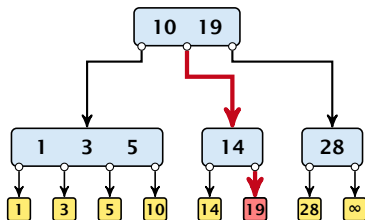
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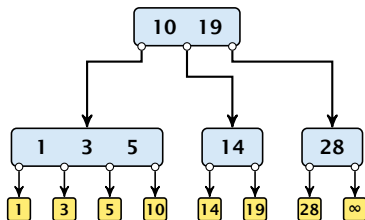
The search is straightforward. It is only important that you need to go all the way to the leaf.

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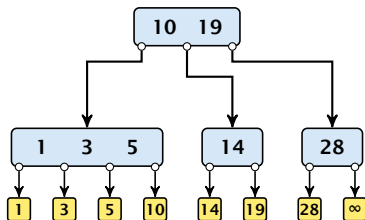
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Time: $\mathcal{O}(b \cdot h) = \mathcal{O}(b \cdot \log n)$, if the individual nodes are organized as linear lists.

Insert

Insert element x:

- ▶ Follow the path as if searching for $\text{key}[x]$.

- ▶ If this search ends in leaf ℓ, insert x before this leaf.

- ▶ For this add $\text{key}[x]$ to the key-list of the last internal node v on the path.

- ▶ If after the insert v contains b nodes, do Rebalance(v).

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Rebalance(v):

▶ Let k_i, $i = 1, \ldots, b$ denote the keys stored in v.

▶ Let $j := \lfloor \frac{b+1}{2} \rfloor$ be the middle element.

▶ Create two nodes v_1, and v_2. v_1 gets all keys $k_1, \ldots, k_{j-1}$ and v_2 gets keys $k_{j+1}, \ldots, k_b$.

▶ Both nodes get at least $\lfloor \frac{b-1}{2} \rfloor$ keys, and have therefore degree at least $\lfloor \frac{b-1}{2} \rfloor + 1 \geq a$ since $b \geq 2a - 1$.

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▶ The key k_j is promoted to the parent of v. The current pointer to v is altered to point to v_1, and a new pointer (to the right of k_j) in the parent is added to point to v_2.

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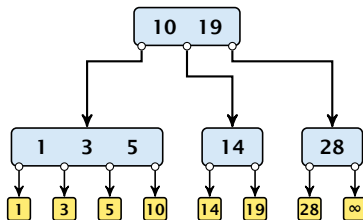
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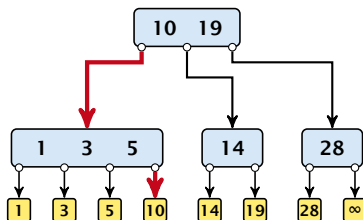
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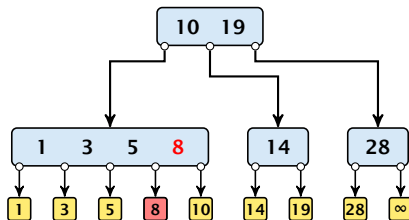
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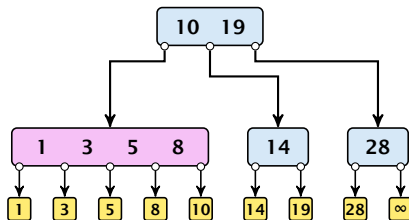
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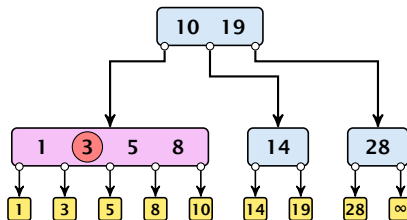
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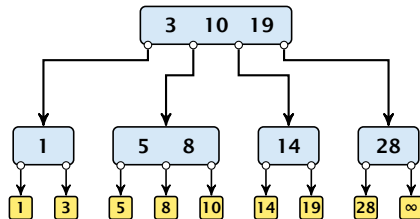
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Insert

Insert

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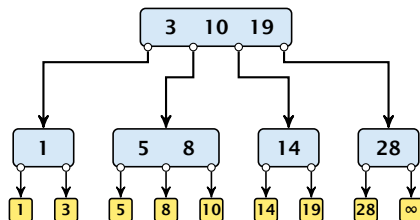
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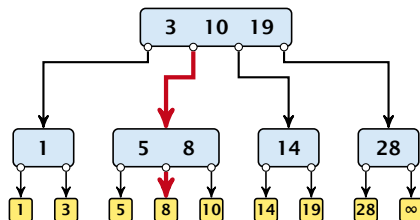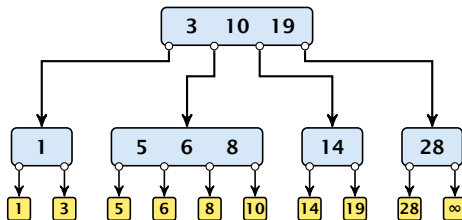
Insert(6)

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Insert(6)

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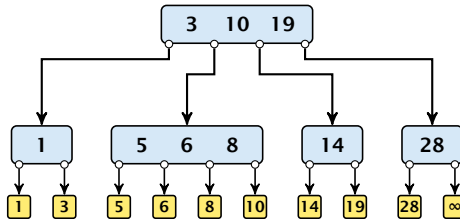
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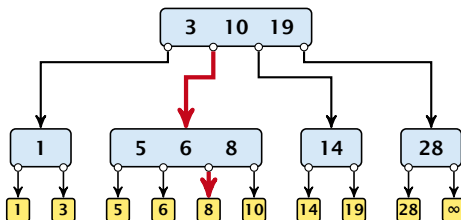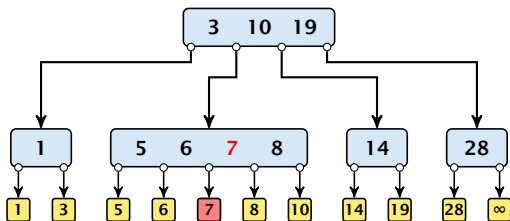
Insert(7)

7.5 (a, b)-trees

Insert

Insert(7)

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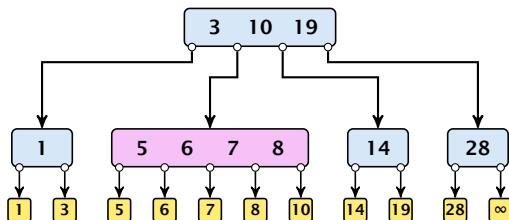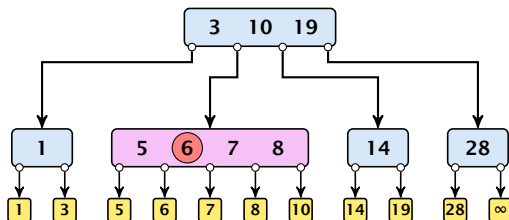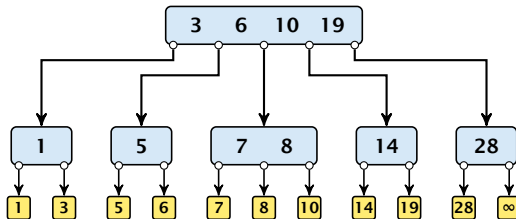
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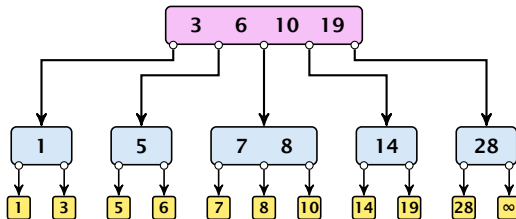
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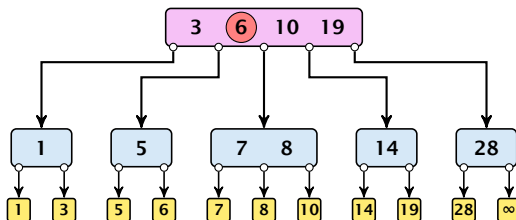
Insert

Insert(7)

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Delete

Delete element x (pointer to leaf vertex):

▶ Let v denote the parent of x. If $\text{key}[x]$ is contained in v, remove the key from v, and delete the leaf vertex.

▶ Otherwise delete the key of the predecessor of x from v; delete the leaf vertex; and replace the occurrence of $\text{key}[x]$ in internal nodes by the predecessor key. (Note that it appears in exactly one internal vertex).

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▶ If there is a neighbour of v that has at least a keys take over the largest (if right neighbor) or smallest (if left neighbour) and the corresponding sub-tree.

▶ If not: merge v with one of its neighbours.

▶ The merged node contains at most $(a-2) + (a-1) + 1$ keys, and has therefore at most $2a - 1 \leq b$ successors.

▶ Then rebalance the parent.

▶ During this process the root may become empty. In this case the root is deleted and the height of the tree decreases.

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Delete(10)
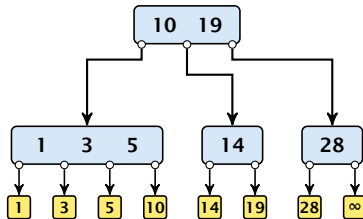
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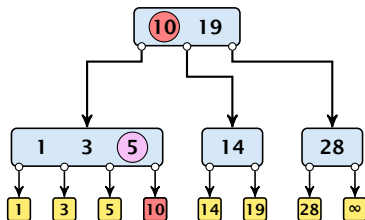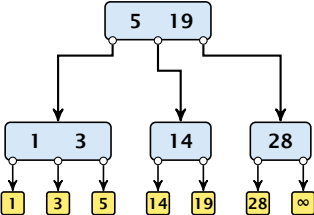
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Delete(10)

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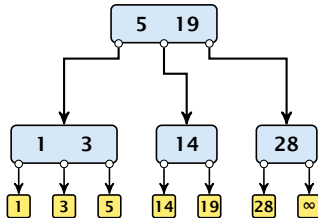
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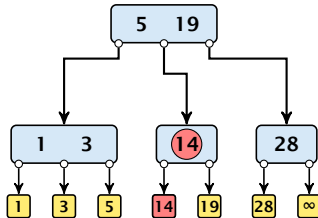
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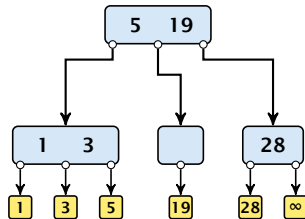
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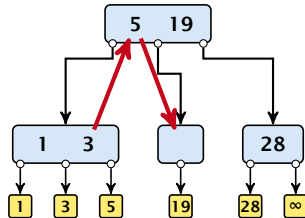
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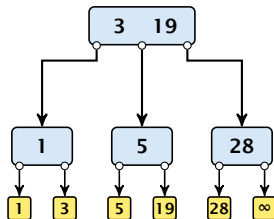
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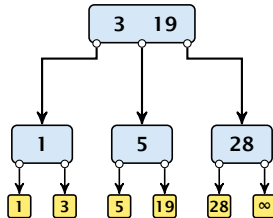
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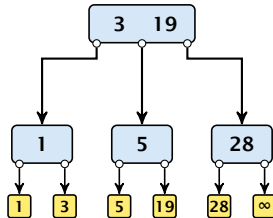
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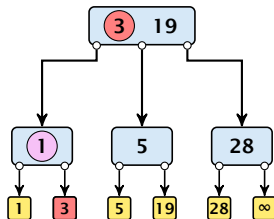
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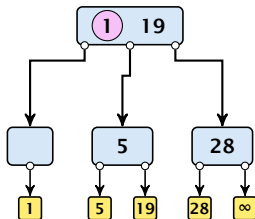
Delete

Delete(3)
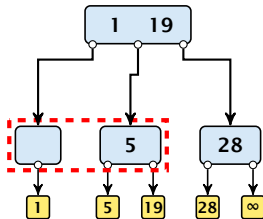
Delete

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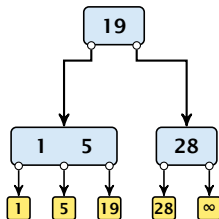
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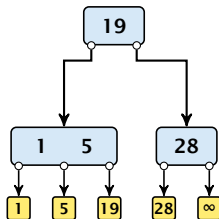
Delete(3)

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Delete(1)
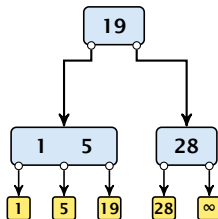
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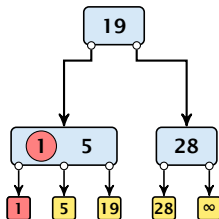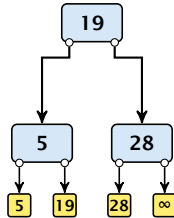
Delete(1)

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Delete(19)
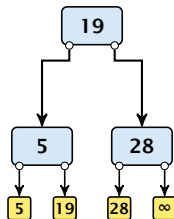
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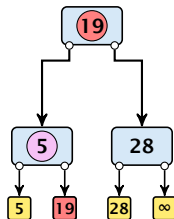
Delete(19)
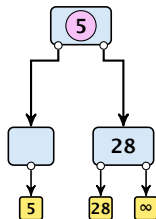
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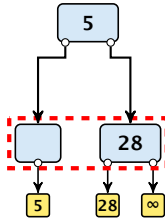
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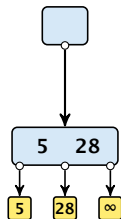
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(2, 4)-trees and red black trees

There is a close relation between red-black trees and $(2, 4)$-trees:

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Note that this correspondence is not unique. In particular, there are different red-black trees that correspond to the same $(2, 4)$-tree.

7.6 Skip Lists

Why do we not use a list for implementing the ADT Dynamic Set?

- time for search $\Theta(n)$
- time for insert $\Theta(n)$ (dominated by searching the item)
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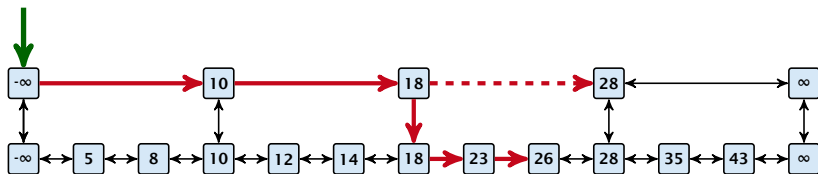
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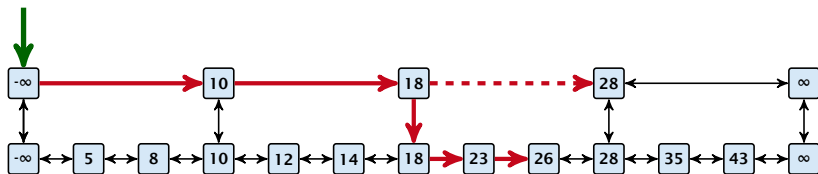
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Choose $|L_1| = \sqrt{n}$. Then search time $\Theta(\sqrt{n})$.

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- ▶ ...

- ▶ At most $|L_k| + \sum_{i=1}^{k} \frac{L_{i-1}}{L_i} + 3(k + 1)$ steps.

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Choosing $k = \Theta(\log n)$ gives a logarithmic running time.

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Use randomization instead!

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▶ If we want that in L_i we always skip over roughly the same number of elements in L_{i-1} an insert or delete may require a lot of re-organisation.

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- A search operation gives you the insert position for element x in every list.

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The time for both operations is dominated by the search time.

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High Probability

Definition 10 (High Probability)

We say a **randomized** algorithm has running time $\mathcal{O}(\log n)$ with high probability if for any constant α the running time is at most $\mathcal{O}(\log n)$ with probability at least $1 - \frac{1}{n^\alpha}$.

Here the $\mathcal{O}$-notation hides a constant that may depend on α.

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High Probability

Suppose there are a polynomially many events $E_1, E_2, \ldots, E_\ell$, $\ell = n^c$ each holding with high probability (e.g. E_i may be the event that the i-th search in a skip list takes time at most $\mathcal{O}(\log n)$).

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This means $\Pr[E_1 \wedge \cdots \wedge E_\ell]$ holds with high probability.

7.6 Skip Lists

Lemma 11

A search (and, hence, also insert and delete) in a skip list with n elements takes time $\mathcal{O}(\log n)$ with high probability (w. h. p.).

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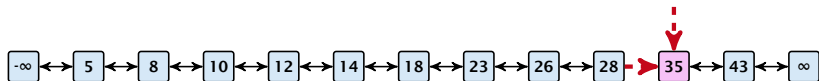
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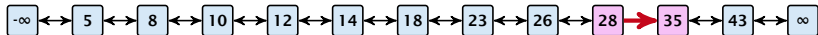
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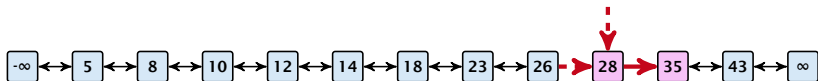
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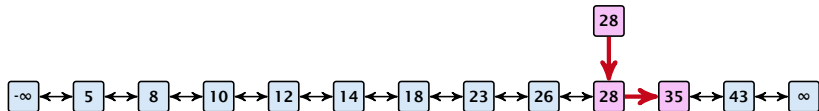
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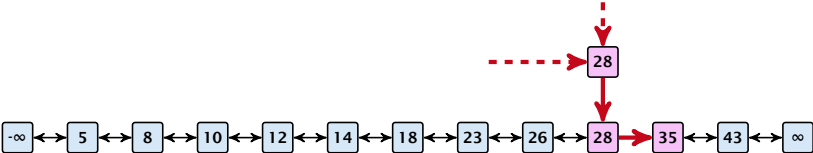
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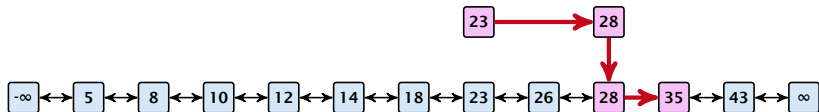
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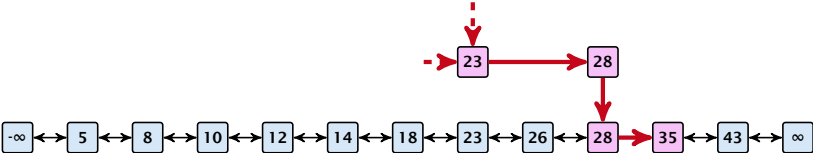
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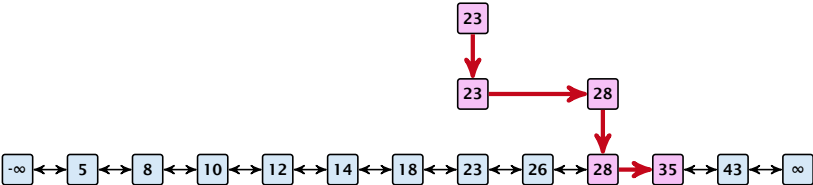
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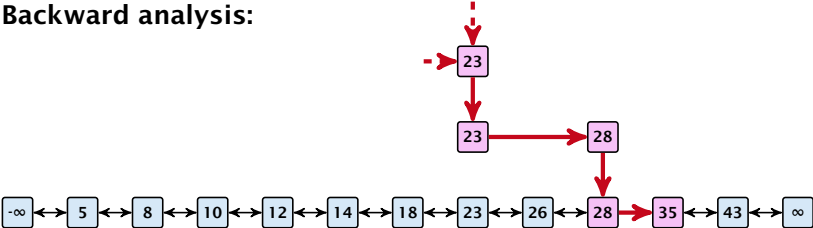
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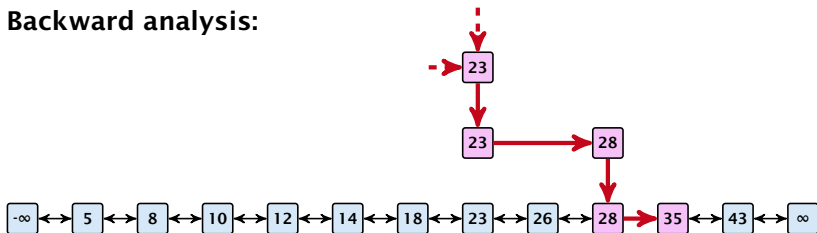
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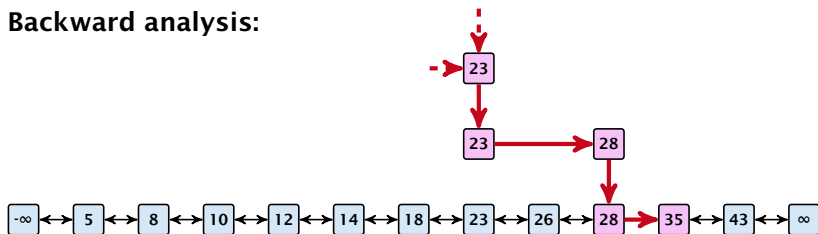
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We show that w.h.p:

▶ A "long" search path must also go very high.

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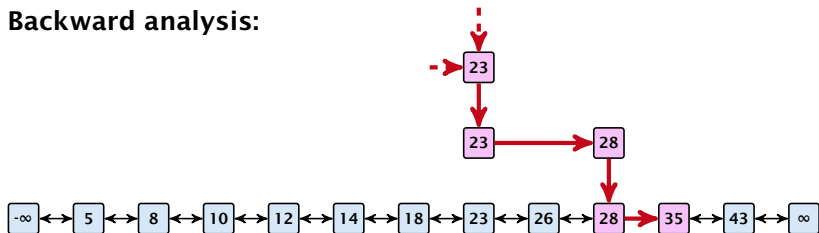
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We show that w.h.p:

▶ A "long" search path must also go very high.

▶ There are no elements in high lists.

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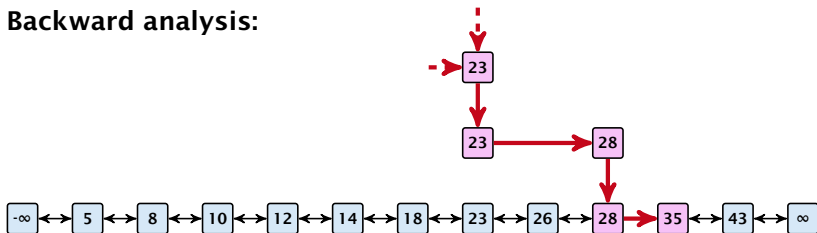
At each point the path goes up with probability $1/2$ and left with probability $1/2$.

We show that w.h.p:

- ▶ A "long" search path must also go very high.
- ▶ There are no elements in high lists.

From this it follows that w.h.p. there are no long paths.

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In particular, this means that during the construction in the backward analysis we see at most k heads (i.e., coin flips that tell you to go up) in z trials.

$\Pr[E_{z,k}]$

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This means, the search requires at most z steps, w. h. p.

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Dictionary:

- ▶ $S.\,\textbf{insert}(x)$: Insert an element x.
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- ▶ $S.\,\textbf{search}(k)$: Return a pointer to an element e with $\text{key}[e] = k$ in S if it exists; otherwise return null.

So far we have implemented the search for a key by carefully choosing split-elements.

Then the memory location of an object x with key k is determined by successively comparing k to split-elements.

Hashing tries to directly compute the memory location from the given key. The goal is to have constant search time.

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Definitions:

▶ Universe U of keys, e.g., $U \subseteq \mathbb{N}_0$. U very large.

▶ Set $S \subseteq U$ of keys, $|S| = m \leq |U|$.

▶ Array $T[0, \ldots, n-1]$ hash-table.

▶ Hash function $h : U \to [0, \ldots, n-1]$.

The hash-function h should fulfill:

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Direct Addressing

Ideally the hash function maps all keys to different memory locations.
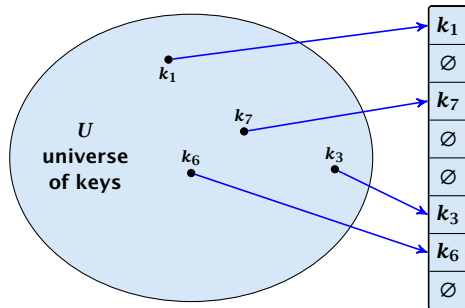
This special case is known as Direct Addressing. It is usually very unrealistic as the universe of keys typically is quite large, and in particular larger than the available memory.

Perfect Hashing

Suppose that we know the set S of actual keys (no insert/no delete). Then we may want to design a simple hash-function that maps all these keys to different memory locations.
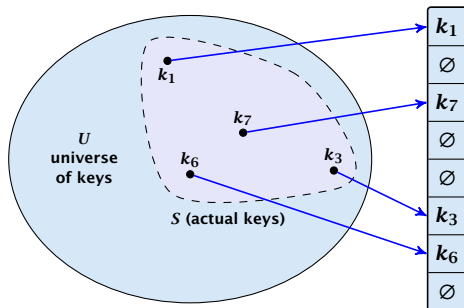
Such a hash function h is called a perfect hash function for set S.

Collisions

If we do not know the keys in advance, the best we can hope for is that the hash function distributes keys evenly across the table.

Problem: Collisions
Usually the universe U is much larger than the table-size n.

Hence, there may be two elements k_1, k_2 from the set S that map to the same memory location (i.e., $h(k_1) = h(k_2)$). This is called a collision.

Collisions

If we do not know the keys in advance, the best we can hope for is that the hash function distributes keys evenly across the table.

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The probability of having a collision when hashing m elements into a table of size n under uniform hashing is at least

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Here the first equality follows since the ℓ-th element that is hashed has a probability of $\frac{n-\ell+1}{n}$ to not generate a collision under the condition that the previous elements did not induce collisions. $\qquad\square$

Collisions
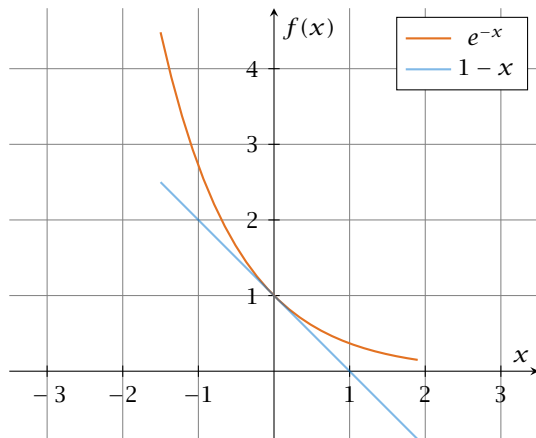
The inequality $1 - x \leq e^{-x}$ is derived by stopping the Taylor-expansion of e^{-x} after the second term.

Resolving Collisions

The methods for dealing with collisions can be classified into the two main types

- **open addressing**, aka. closed hashing
- **hashing with chaining**, aka. closed addressing, open hashing.

There are applications e.g. computer chess where you do not resolve collisions at all.

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Hashing with Chaining

Arrange elements that map to the same position in a linear list.

▶ Access: compute $h(x)$ and search list for $\text{key}[x]$.

▶ Insert: insert at the front of the list.

Hashing with Chaining

Let A denote a strategy for resolving collisions. We use the following notation:

- A^+ denotes the average time for a **successful** search when using A;

- A^- denotes the average time for an **unsuccessful** search when using A;

- We parameterize the complexity results in terms of $\alpha := \frac{m}{n}$, the so-called fill factor of the hash-table.

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$$A^- = 1 + \alpha \ .$$

Hashing with Chaining

For a successful search observe that we do **not** choose a list at random, but we consider a random key k in the hash-table and ask for the search-time for k.

This is 1 plus the number of elements that lie before k in k's list.

Let k_ℓ denote the ℓ-th key inserted into the table.

Let for two keys k_i and k_j, X_{ij} denote the indicator variable for the event that k_i and k_j hash to the same position. Clearly, $\Pr[X_{ij} = 1] = 1/n$ for uniform hashing.

The expected successful search cost is

$$\mathrm{E}\left[\frac{1}{m}\sum_{i=1}^{m}\left(1 + \sum_{j=i+1}^{m} X_{ij}\right)\right]$$

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Hence, the expected cost for a successful search is $A^+ \le 1 + \frac{\alpha}{2}$.

Hashing with Chaining

Disadvantages:

▶ pointers increase memory requirements

▶ pointers may lead to bad cache efficiency

Advantages:

▶ no à priori limit on the number of elements

▶ deletion can be implemented efficiently

▶ by using balanced trees instead of linked list one can also obtain worst-case guarantees.

Open Addressing

All objects are stored in the table itself.

Define a function $h(k, j)$ that determines the table-position to be examined in the j-th step. The values $h(k, 0), \ldots, h(k, n-1)$ must form a permutation of $0, \ldots, n-1$.

Search(k): Try position $h(k, 0)$; if it is empty your search fails; otw. continue with $h(k, 1), h(k, 2), \ldots$.

Insert(x): Search until you find an empty slot; insert your element there. If your search reaches $h(k, n-1)$, and this slot is non-empty then your table is full.

7.7 Hashing

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Open Addressing

Choices for $h(k, j)$:

▶ Linear probing:
$h(k, i) = h(k) + i \mod n$
(sometimes: $h(k, i) = h(k) + ci \mod n$).

▶ Quadratic probing:
$h(k, i) = h(k) + c_1 i + c_2 i^2 \mod n$.

▶ Double hashing:
$h(k, i) = h_1(k) + i h_2(k) \mod n$.

For quadratic probing and double hashing one has to ensure that the search covers all positions in the table (i.e., for double hashing $h_2(k)$ must be relatively prime to n (teilerfremd); for quadratic probing c_1 and c_2 have to be chosen carefully).

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Linear Probing

- ▶ Advantage: Cache-efficiency. The new probe position is very likely to be in the cache.

- ▶ Disadvantage: Primary clustering. Long sequences of occupied table-positions get longer as they have a larger probability to be hit. Furthermore, they can merge forming larger sequences.

Lemma 13

Let L be the method of linear probing for resolving collisions:

$$L^+ \approx \frac{1}{2}\left(1 + \frac{1}{1 - \alpha}\right)$$

$$L^- \approx \frac{1}{2}\left(1 + \frac{1}{(1 - \alpha)^2}\right)$$

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Quadratic Probing

- ▶ Not as cache-efficient as Linear Probing.
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Lemma 14

Let Q be the method of quadratic probing for resolving collisions:

$$Q^+ \approx 1 + \ln\left(\frac{1}{1-\alpha}\right) - \frac{\alpha}{2}$$

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Double Hashing

▶ Any probe into the hash-table usually creates a cache-miss.

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Lemma 15

Let A be the method of double hashing for resolving collisions:

$$D^+ \approx \frac{1}{\alpha} \ln\left(\frac{1}{1-\alpha}\right)$$

$$D^- \approx \frac{1}{1-\alpha}$$

Open Addressing

Some values:

α	Linear Probing		Quadratic Probing		Double Hashing	
	L^+	L^-	Q^+	Q^-	D^+	D^-
0.5	1.5	2.5	1.44	2.19	1.39	2
0.9	5.5	50.5	2.85	11.40	2.55	10
0.95	10.5	200.5	3.52	22.05	3.15	20

Open Addressing
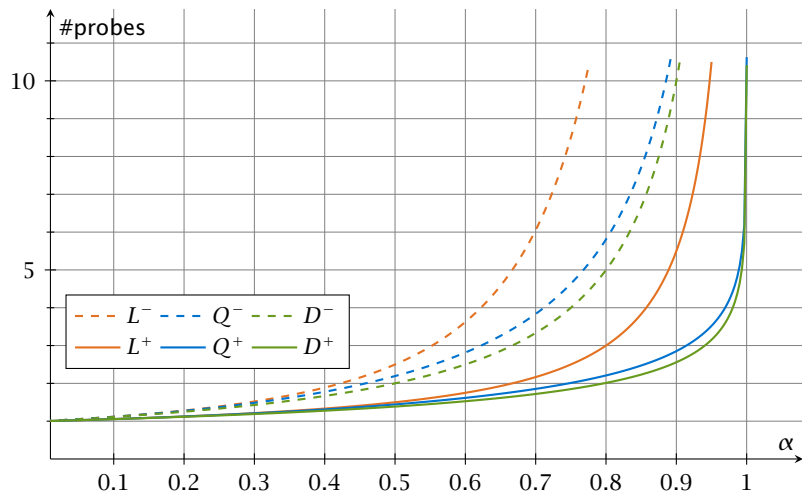
Analysis of Idealized Open Address Hashing

We analyze the time for a search in a very idealized Open Addressing scheme.

- The probe sequence $h(k,0), h(k,1), h(k,2), \ldots$ is equally likely to be any permutation of $\langle 0, 1, \ldots, n-1 \rangle$.

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Let A_i denote the event that the i-th probe **occurs** and is to a non-empty slot.

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$$= \Pr[A_1] \cdot \Pr[A_2 \mid A_1] \cdot \Pr[A_3 \mid A_1 \cap A_2] \cdot$$
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$$\leq \left(\frac{m}{n}\right)^{i-1} = \alpha^{i-1} \ .$$

$\mathrm{E}[X]$

Analysis of Idealized Open Address Hashing

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Analysis of Idealized Open Address Hashing

$$E[X] = \sum_{i=1}^{\infty} \Pr[X \geq i] \leq \sum_{i=1}^{\infty} \alpha^{i-1} = \sum_{i=0}^{\infty} \alpha^i = \frac{1}{1-\alpha} \ .$$

$$\frac{1}{1-\alpha} = 1 + \alpha + \alpha^2 + \alpha^3 + \dots$$

Analysis of Idealized Open Address Hashing

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$i = 1$

$$\sum_i i \Pr[X = i] = \sum_i \Pr[X \geq i]$$

Analysis of Idealized Open Address Hashing

$i = 2$

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Analysis of Idealized Open Address Hashing

$i = 3$

Analysis of Idealized Open Address Hashing

$i = 4$
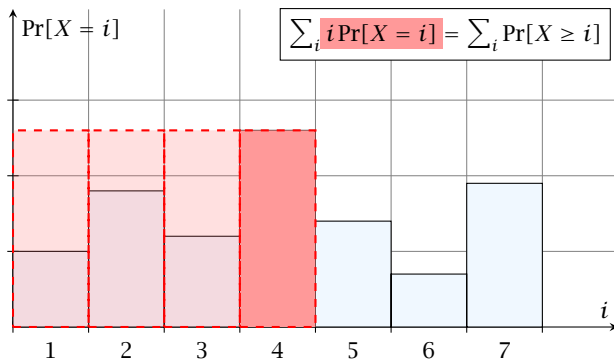
Analysis of Idealized Open Address Hashing

$i = 1$
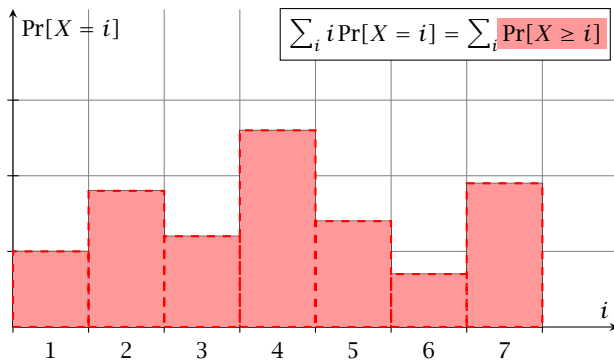
Analysis of Idealized Open Address Hashing

$i = 2$

Analysis of Idealized Open Address Hashing
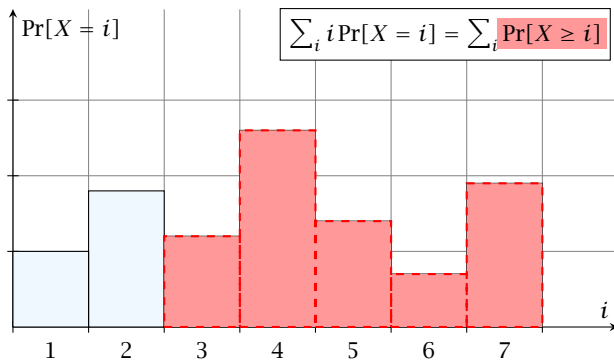
$i = 3$

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Analysis of Idealized Open Address Hashing

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Analysis of Idealized Open Address Hashing
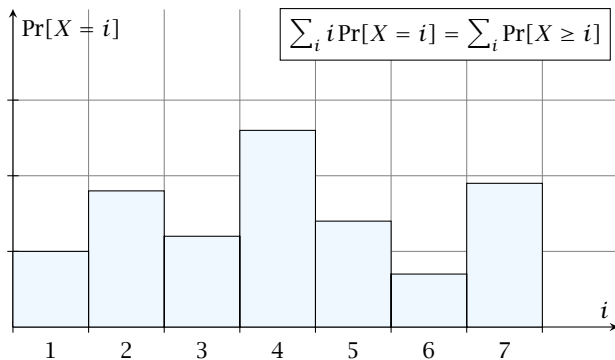
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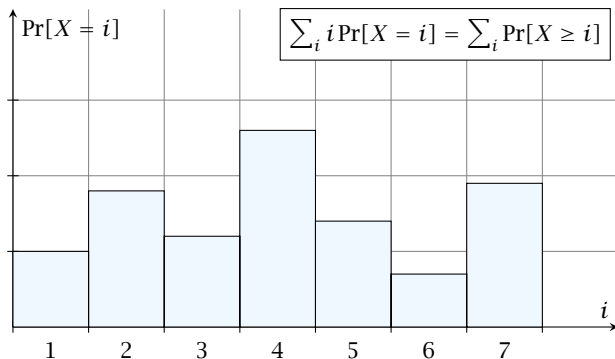
The j-th rectangle appears in both sums j times. (j times in the first due to multiplication with j; and j times in the second for summands $i = 1, 2, \ldots, j$)

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Analysis of Idealized Open Address Hashing

$$\sum_{k=m-n+1}^{n} \frac{1}{k} \le \int_{m-n}^{n} \frac{1}{x}\, dx$$

Deletions in Hashtables

How do we delete in a hash-table?

▶ For hashing with chaining this is not a problem. Simply search for the key, and delete the item in the corresponding list.

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Deletions in Hashtables

- Simply removing a key might interrupt the probe sequence of other keys which then cannot be found anymore.

- One can delete an element by replacing it with a deleted-marker.

  - During an insertion if a deleted-marker is encountered an element can be inserted there.
  - During a search a deleted-marker must not be used to terminate the probe sequence.

- The table could fill up with deleted-markers leading to bad performance.

- If a table contains many deleted-markers (linear fraction of the keys) one can rehash the whole table and amortize the cost for this rehash against the cost for the deletions.

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Algorithm 12 delete(p)
1: $T[p] \leftarrow$ null
2: $p \leftarrow \text{succ}(p)$
3: **while** $T[p] \neq$ null **do**
4: $\quad\quad y \leftarrow T[p]$
5: $\quad\quad T[p] \leftarrow$ null
6: $\quad\quad p \leftarrow \text{succ}(p)$
7: $\quad\quad \text{insert}(y)$

p is the index into the table-cell that contains the object to be deleted.

Pointers into the hash-table become invalid.

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Universal Hashing

Regardless, of the choice of hash-function there is always an input (a set of keys) that has a very poor worst-case behaviour.

Therefore, so far we assumed that the hash-function is random so that regardless of the input the average case behaviour is good.

However, the assumption of uniform hashing that h is chosen randomly from all functions $f : U \to [0, \ldots, n-1]$ is clearly unrealistic as there are $n^{|U|}$ such functions. Even writing down such a function would take $|U| \log n$ bits.

Universal hashing tries to define a set $\mathcal{H}$ of functions that is much smaller but still leads to good average case behaviour when selecting a hash-function uniformly at random from $\mathcal{H}$.

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Definition 16

A class $\mathcal{H}$ of hash-functions from the universe U into the set $\{0, \ldots, n-1\}$ is called universal if for all $u_1, u_2 \in U$ with $u_1 \neq u_2$

$$\Pr[h(u_1) = h(u_2)] \leq \frac{1}{n} \ ,$$

where the probability is w. r. t. the choice of a random hash-function from set $\mathcal{H}$.

Note that this means that the probability of a collision between two arbitrary elements is at most $\frac{1}{n}$.

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A class $\mathcal{H}$ of hash-functions from the universe U into the set $\{0, \ldots, n-1\}$ is called 2-independent (pairwise independent) if the following two conditions hold

- ▸ For any key $u \in U$, and $t \in \{0, \ldots, n-1\}$ $\Pr[h(u) = t] = \frac{1}{n}$, i.e., a key is distributed uniformly within the hash-table.

- ▸ For all $u_1, u_2 \in U$ with $u_1 \neq u_2$, and for any two hash-positions t_1, t_2:

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Universal Hashing

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Universal Hashing

Let $U := \{0, \ldots, p-1\}$ for a prime p. Let $\mathbb{Z}_p := \{0, \ldots, p-1\}$, and let $\mathbb{Z}_p^* := \{1, \ldots, p-1\}$ denote the set of invertible elements in $\mathbb{Z}_p$.

Define

$$h_{a,b}(x) := (ax + b \bmod p) \bmod n$$

Lemma 20

The class

$$\mathcal{H} = \{h_{a,b} \mid a \in \mathbb{Z}_p^*, b \in \mathbb{Z}_p\}$$

is a universal class of hash-functions from U to $\{0, \ldots, n-1\}$.

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▶ $ax + b \not\equiv ay + b \pmod{p}$

If $x \neq y$ then $(x - y) \not\equiv 0 \pmod{p}$.

Multiplying with $a \not\equiv 0 \pmod{p}$ gives

$$a(x - y) \not\equiv 0 \pmod{p}$$

where we use that $\mathbb{Z}_p$ is a field (Körper) and, hence, has no zero divisors (nullteilerfrei).

Universal Hashing

Proof.

Let $x, y \in U$ be two distinct keys. We have to show that the probability of a collision is only $1/n$.

▶ $ax + b \not\equiv ay + b \pmod{p}$

If $x \neq y$ then $(x - y) \not\equiv 0 \pmod{p}$.

Multiplying with $a \not\equiv 0 \pmod{p}$ gives

$$a(x - y) \not\equiv 0 \pmod{p}$$

where we use that $\mathbb{Z}_p$ is a field (Körper) and, hence, has no zero divisors (nullteilerfrei).

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▶ The hash-function does not generate collisions before the $(\bmod\ n)$-operation. Furthermore, every choice (a, b) is mapped to a different pair (t_x, t_y) with $t_x := ax + b$ and $t_y := ay + b$.

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$$a \equiv (t_x - t_y)(x - y)^{-1} \qquad (\bmod\ p)$$
$$b \equiv t_y - ay \qquad (\bmod\ p)$$

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There is a one-to-one correspondence between hash-functions (pairs (a, b), $a \neq 0$) and pairs (t_x, t_y), $t_x \neq t_y$.

Therefore, we can view the first step (before the $\mod n$-operation) as choosing a pair (t_x, t_y), $t_x \neq t_y$ uniformly at random.

What happens when we do the $\mod n$ operation?

Fix a value t_x. There are $p - 1$ possible values for choosing t_y.

From the range $0, \ldots, p - 1$ the values $t_x, t_x + n, t_x + 2n, \ldots$ map to t_x after the modulo-operation. These are at most $\lceil p/n \rceil$ values.

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As $t_y \neq t_x$ there are

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possibilities for choosing t_y such that the final hash-value creates a collision.

This happens with probability at most $\frac{1}{n}$.

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As $t_y \neq t_x$ there are

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It is also possible to show that $\mathcal{H}$ is an (almost) pairwise independent class of hash-functions.

$$\Pr_{t_x \neq t_y \in \mathbb{Z}_p^2} \left[\begin{array}{c} t_x \bmod n = h_1 \\ \wedge \\ t_y \bmod n = h_2 \end{array} \right]$$

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$$\frac{\left\lfloor \frac{p}{n} \right\rfloor^2}{p(p-1)} \leq \Pr_{t_x \neq t_y \in \mathbb{Z}_p^2} \left[\begin{array}{c} t_x \bmod n = h_1 \\ \wedge \\ t_y \bmod n = h_2 \end{array} \right] \leq \frac{\left\lceil \frac{p}{n} \right\rceil^2}{p(p-1)}$$

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Note that the middle is the probability that $h(x) = h_1$ and $h(y) = h_2$. The total number of choices for (t_x, t_y) is $p(p-1)$. The number of choices for t_x (t_y) such that $t_x \bmod n = h_1$ ($t_y \bmod n = h_2$) lies between $\lfloor \frac{p}{n} \rfloor$ and $\lceil \frac{p}{n} \rceil$.

Universal Hashing

Definition 21

Let $d \in \mathbb{N}$; $q \geq (d+1)n$ be a prime; and let $\bar{a} \in \{0, \ldots, q-1\}^{d+1}$. Define for $x \in \{0, \ldots, q-1\}$

$$h_{\bar{a}}(x) := \Big(\sum_{i=0}^{d} a_i x^i \bmod q \Big) \bmod n \ .$$

Let $\mathcal{H}_n^d := \{ h_{\bar{a}} \mid \bar{a} \in \{0, \ldots, q-1\}^{d+1} \}$. The class $\mathcal{H}_n^d$ is $(e, d+1)$-independent.

Note that in the previous case we had $d = 1$ and chose $a_d \neq 0$.

Universal Hashing

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For the coefficients $\bar{a} \in \{0, \ldots, q-1\}^{d+1}$ let $f_{\bar{a}}$ denote the polynomial

$$f_{\bar{a}}(x) = \Big(\sum_{i=0}^{d} a_i x^i \Big) \bmod q$$

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Fix $\ell \leq d+1$; let $x_1, \ldots, x_\ell \in \{0, \ldots, q-1\}$ be keys, and let $t_1, \ldots, t_\ell$ denote the corresponding hash-function values.

Let $A^\ell = \{h_{\vec{a}} \in \mathcal{H} \mid h_{\vec{a}}(x_i) = t_i \text{ for all } i \in \{1, \ldots, \ell\}\}$
Then
$$h_{\vec{a}} \in A^\ell \Leftrightarrow h_{\vec{a}} = f_{\vec{a}} \bmod n \text{ and}$$

$$f_{\vec{a}}(x_i) \in \underbrace{\{t_i + \alpha \cdot n \mid \alpha \in \{0, \ldots, \lceil \frac{q}{n} \rceil - 1\}\}}_{=: B_i}$$

In order to obtain the cardinality of A^ℓ we choose our polynomial by fixing $d+1$ points.

We first fix the values for inputs $x_1, \ldots, x_\ell$.
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$$|B_1| \cdot \ldots \cdot |B_\ell|$$

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This shows that the $\mathcal{H}$ is $(e, d+1)$-universal.

The last step followed from $q \ge (d+1)n$, and $\ell \le d+1$.

Perfect Hashing

Suppose that we **know** the set S of actual keys (no insert/no delete). Then we may want to design a **simple** hash-function that maps all these keys to different memory locations.

Perfect Hashing

Let $m = |S|$. We could simply choose the hash-table size very large so that we don't get any collisions.

Using a universal hash-function the expected number of collisions is

$$E[\#\text{Collisions}] = \binom{m}{2} \cdot \frac{1}{n} \ .$$

If we choose $n = m^2$ the expected number of collisions is strictly less than $\frac{1}{2}$.

Can we get an upper bound on the probability of having collisions?

The probability of having 1 or more collisions can be at most $\frac{1}{2}$ as otherwise the expectation would be larger than $\frac{1}{2}$.

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We can find such a hash-function by a few trials.

However, a hash-table size of $n = m^2$ is very very high.

We construct a two-level scheme. We first use a hash-function that maps elements from S to m buckets.

Let m_j denote the number of items that are hashed to the j-th bucket. For each bucket we choose a second hash-function that maps the elements of the bucket into a table of size m_j^2. The second function can be chosen such that all elements are mapped to different locations.

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$$= 2\binom{m}{2}\frac{1}{m} + m = 2m - 1 \ .$$

Perfect Hashing

We need only $\mathcal{O}(m)$ time to construct a hash-function h with $\sum_j m_j^2 = \mathcal{O}(4m)$, because with probability at least $1/2$ a random function from a universal family will have this property.

Then we construct a hash-table h_j for every bucket. This takes expected time $\mathcal{O}(m_j)$ for every bucket. A random function h_j is collision-free with probability at least $1/2$. We need $\mathcal{O}(m_j)$ to test this.

We only need that the hash-functions are chosen from a universal family!!!

Cuckoo Hashing

Goal:

Try to generate a hash-table with constant worst-case search time in a dynamic scenario.

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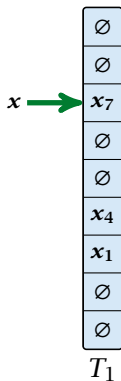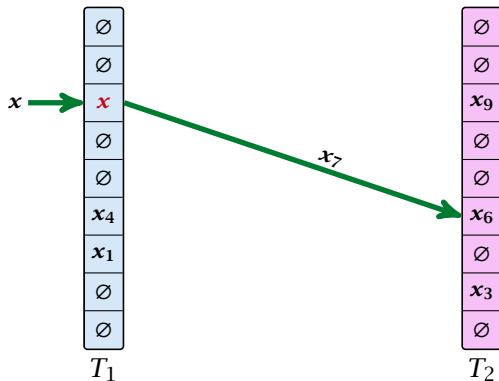
Cuckoo Hashing

Insert:

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Cuckoo Hashing

Algorithm 13 Cuckoo-Insert(x)

1: **if** $T_1[h_1(x)] = x \lor T_2[h_2(x)] = x$ **then return**
2: steps $\leftarrow$ 1
3: **while** steps $\leq$ maxsteps **do**
4:      exchange x and $T_1[h_1(x)]$
5:      **if** x = null **then return**
6:      exchange x and $T_2[h_2(x)]$
7:      **if** x = null **then return**
8:      steps $\leftarrow$ steps $+1$
9: rehash() // change hash-functions; rehash everything
10: Cuckoo-Insert(x)

Cuckoo Hashing

▶ We call one iteration through the while-loop a step of the algorithm.

▶ We call a sequence of iterations through the while-loop without the termination condition becoming true a phase of the algorithm.

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What is the expected time for an insert-operation?

We first analyze the probability that we end-up in an infinite loop (that is then terminated after maxsteps steps).

Formally what is the probability to enter an infinite loop that touches s different keys?

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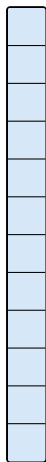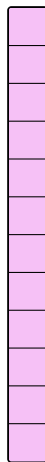We first analyze the probability that we end-up in an infinite loop (that is then terminated after maxsteps steps).

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Cuckoo Hashing: Insert

T_1 T_2

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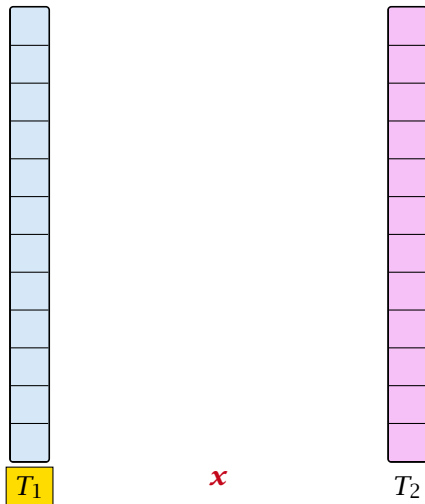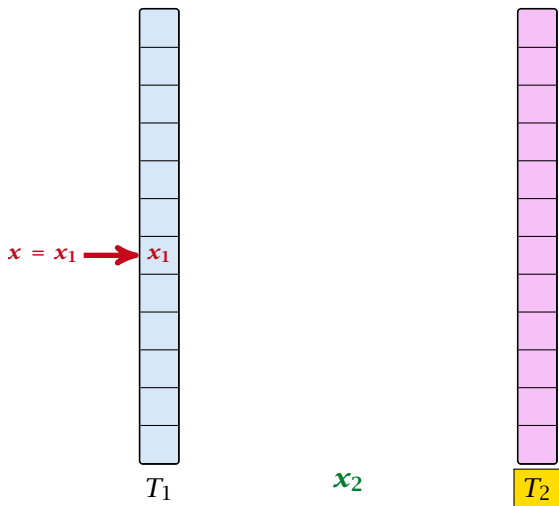$x = x_1$ x_1

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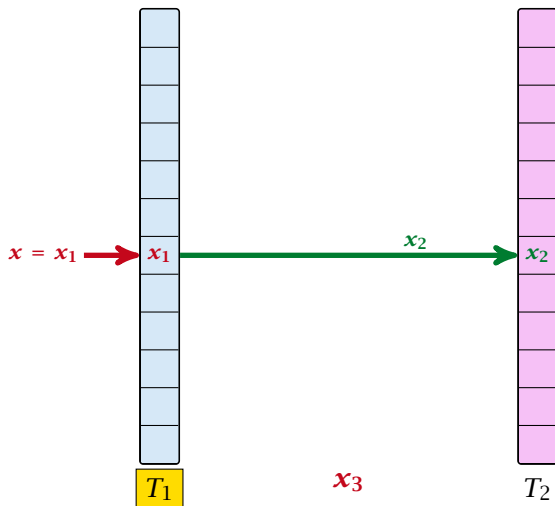# Cuckoo Hashing: Insert

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A cycle-structure is active if for every key x_ℓ (linking a cell p_i from T_1 and a cell p_j from T_2) we have

$$h_1(x_\ell) = p_i \qquad \text{and} \qquad h_2(x_\ell) = p_j$$

Observation:
If during a phase the insert-procedure runs into a cycle there must exist an active cycle structure of size $s \geq 3$.

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What is the probability that all keys in a cycle-structure of size s correctly map into their T_1-cell?

This probability is at most $\frac{\mu}{n^s}$ since h_1 is a (μ, s)-independent hash-function.

What is the probability that all keys in the cycle-structure of size s correctly map into their T_2-cell?

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▶ There are at most s^2 possibilities where to attach the forward and backward links.

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Hence,

$$\Pr[\text{cycle}] = \mathcal{O}\left(\frac{1}{m^2}\right) \ .$$

Cuckoo Hashing

Now, we analyze the probability that a phase is not successful without running into a closed cycle.

Cuckoo Hashing
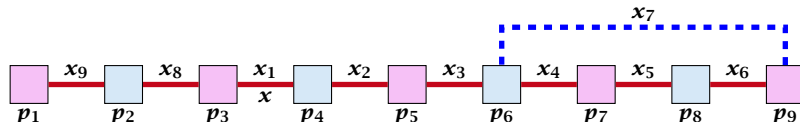
Sequence of visited keys:

$x = x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_3, x_2, x_1 = x, x_8, x_9, \ldots$

Cuckoo Hashing

Consider the sequence of not necessarily distinct keys starting with x in the order that they are visited during the phase.

Lemma 22
If the sequence is of length p then there exists a sub-sequence of at least $\frac{p+2}{3}$ keys starting with x of distinct keys.

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If the sequence is of length p then there exists a sub-sequence of at least $\frac{p+2}{3}$ keys starting with x of distinct keys.

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Proof.
Let i be the number of keys (including x) that we see before the first repeated key. Let j denote the total number of distinct keys.

The sequence is of the form:
$$x = x_1 \to x_2 \to \cdots \to x_i \to x_r \to x_{r-1} \to \cdots \to x_1 \to x_{i+1} \to \cdots \to x_j$$

As $r \le i - 1$ the length p of the sequence is

$$p = i + r + (j - i) \le i + j - 1 \ .$$

Either sub-sequence $x_1 \to x_2 \to \cdots \to x_i$ or sub-sequence $x_1 \to x_{i+1} \to \cdots \to x_j$ has at least $\frac{p+2}{3}$ elements.

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Observation:
If a phase takes at least t steps without running into a cycle there must exist an active path-structure of size $(2t + 2)/3$.

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This gives $\text{maxsteps} = \Theta(\log m)$.

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So far we estimated

$$\Pr[\text{cycle}] \leq \mathcal{O}\Big(\frac{1}{m^2}\Big)$$

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This means the expected cost for a successful phase is constant (even after accounting for the cost of the incomplete step that finishes the phase).

Cuckoo Hashing

A phase that is not successful induces cost for doing a complete rehash (this dominates the cost for the steps in the phase).

The probability that a phase is not successful is $p = \mathcal{O}(1/m^2)$ (probability $\mathcal{O}(1/m^2)$ of running into a cycle and probability $\mathcal{O}(1/m^2)$ of reaching maxsteps without running into a cycle).

A rehash try requires m insertions and takes expected constant time per insertion. It fails with probability $p := \mathcal{O}(1/m)$.

The expected number of unsuccessful rehashes is $\sum_{i \geq 1} p^i = \frac{1}{1-p} - 1 = \frac{p}{1-p} = \mathcal{O}(p)$.

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The expected cost for all rehashes is

$$\mathrm{E}\left[\sum_i \sum_s Z_i X_i^s\right]$$

Note that Z_i is independent of X_j^s, $j \geq i$ (however, it is not independent of X_j^s, $j < i$). Hence,

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Cuckoo Hashing

What kind of hash-functions do we need?

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How do we make sure that $n \geq (1 + \epsilon)m$?

▸ Let $\alpha := 1/(1 + \epsilon)$.

▸ Keep track of the number of elements in the table. When $m \geq \alpha n$ we double n and do a complete re-hash (table-expand).

▸ Whenever m drops below $\alpha n/4$ we divide n by 2 and do a rehash (table-shrink).

▸ Note that right after a change in table-size we have $m = \alpha n/2$. In order for a table-expand to occur at least $\alpha n/2$ insertions are required. Similar, for a table-shrink at least $\alpha n/4$ deletions must occur.

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8 Priority Queues

A Priority Queue S is a dynamic set data structure that supports the following operations:

- ▶ $S.\text{build}(x_1, \ldots, x_n)$: Creates a data-structure that contains just the elements $x_1, \ldots, x_n$.

- ▶ $S.\text{insert}(x)$: Adds element x to the data-structure.

- ▶ element $S.\text{minimum}()$: Returns an element $x \in S$ with minimum key-value $\text{key}[x]$.

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▸ **handle** $S.$ **insert**(x)**:** Adds element x to the data-structure, and returns a handle to the object for future reference.

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Dijkstra's Shortest Path Algorithm

Algorithm 14 Shortest-Path($G = (V, E, d), s \in V$)

1: **Input:** weighted graph $G = (V, E, d)$; start vertex s;
2: **Output:** key-field of every node contains distance from s;
3: S.build(); // build empty priority queue
4: **for all** $v \in V \setminus \{s\}$ **do**
5:     v.key $\leftarrow \infty$;
6:     $h_v \leftarrow S$.insert(v);
7: s.key $\leftarrow 0$; S.insert(s);
8: **while** S.is-empty() = false **do**
9:     $v \leftarrow S$.delete-min();
10:     **for all** $x \in V$ s.t. $(v, x) \in E$ **do**
11:         **if** x.key $> v$.key $+ d(v, x)$ **then**
12:             S.decrease-key(h_x, v.key $+ d(v, x)$);
13:             x.key $\leftarrow v$.key $+ d(v, x)$;

Prim's Minimum Spanning Tree Algorithm

Algorithm 15 Prim-MST($G = (V, E, d), s \in V$)

1: **Input:** weighted graph $G = (V, E, d)$; start vertex s;
2: **Output:** pred-fields encode MST;
3: S.build(); // build empty priority queue
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14:              $x.\text{pred} \leftarrow v$;

Both algorithms require:

- 1 build() operation
- $|V|$ insert() operations
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How good a running time can we obtain?

Analysis of Dijkstra and Prim

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Operation	Binary Heap	BST	Binomial Heap	Fibonacci Heap*
build	n	$n \log n$	$n \log n$	n
minimum	1	$\log n$	$\log n$	1
is-empty	1	1	1	1
insert	$\log n$	$\log n$	$\log n$	1
delete	$\log n$**	$\log n$	$\log n$	$\log n$
delete-min	$\log n$	$\log n$	$\log n$	$\log n$
decrease-key	$\log n$	$\log n$	$\log n$	1
merge	n	$n \log n$	$\log n$	1

Note that most applications use **build()** only to create an empty heap which then costs time 1.

The standard version of binary heaps is not addressable, and hence does not support a delete operation.

Fibonacci heaps only give an amortized guarantee.

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Using Binary Heaps, Prim and Dijkstra run in time
$\mathcal{O}((|V| + |E|) \log |V|)$.

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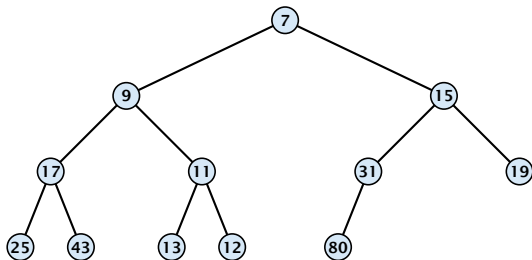
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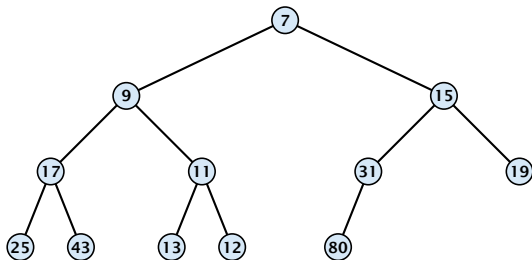
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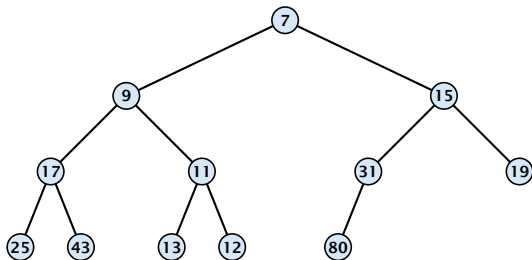
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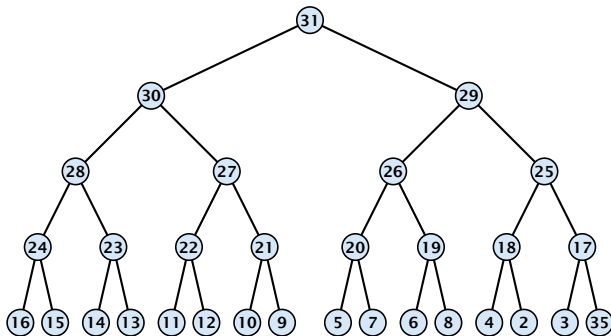▶ is-empty(): check whether root-pointer is null. Time $\mathcal{O}(1)$.

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2. Exchange with parent until heap property is fulfilled.

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- ▸ **delete(h)**: swap with x and bubble up or sift-down. Time $\mathcal{O}(\log n)$.

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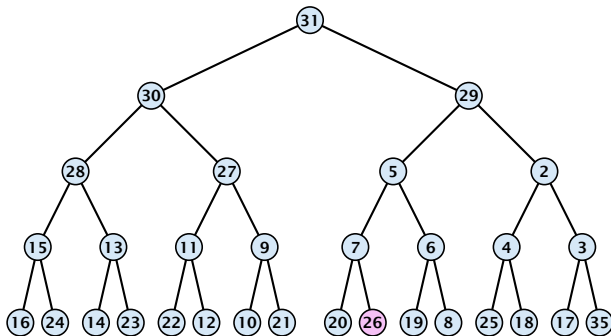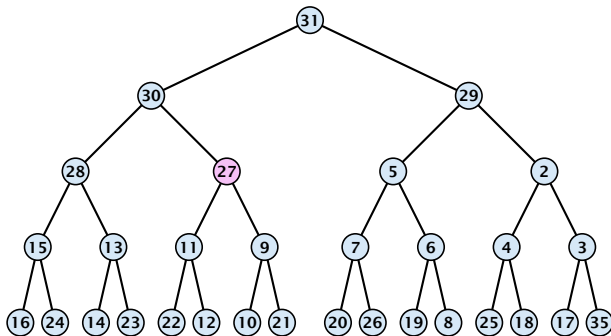$$\sum_{\text{levels } \ell} 2^\ell \cdot (h - \ell) = \mathcal{O}(2^h) = \mathcal{O}(n)$$
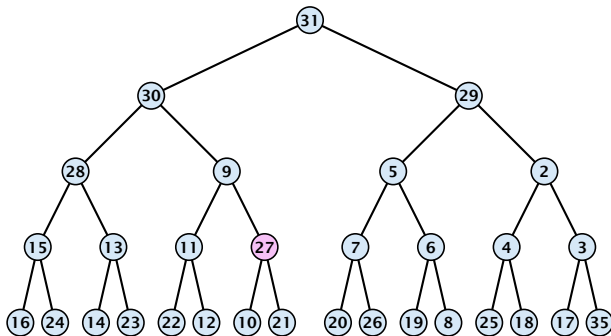
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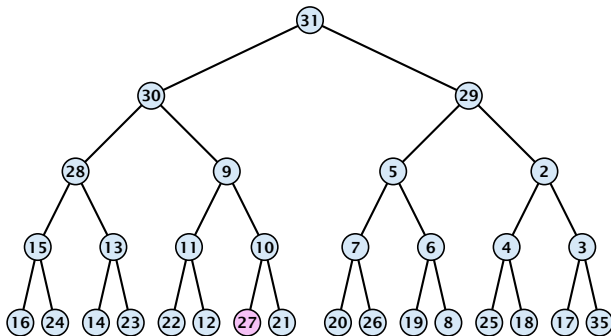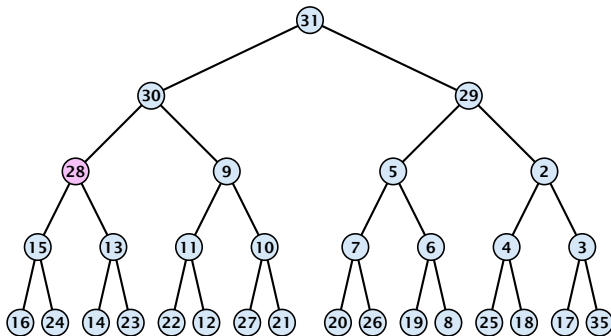$$\sum_{\text{levels } \ell} 2^{\ell} \cdot (h - \ell) = \mathcal{O}(2^h) = \mathcal{O}(n)$$

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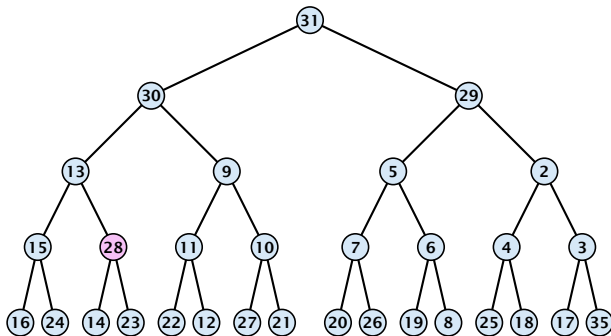
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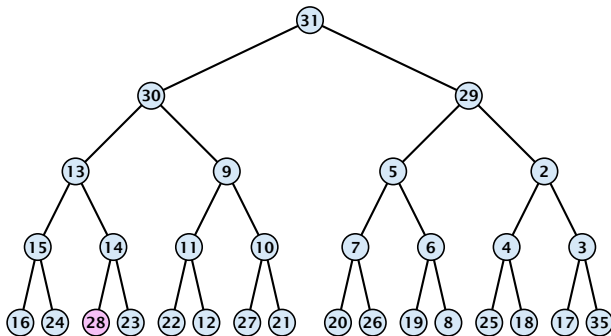
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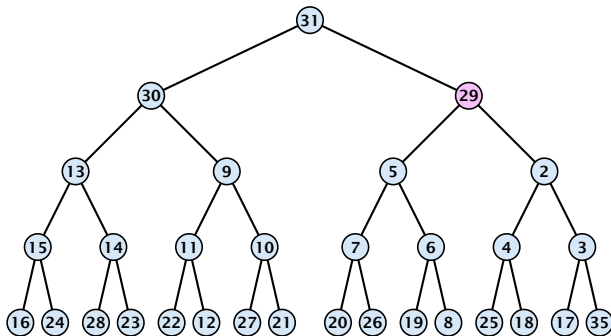$$\sum_{\text{levels } \ell} 2^{\ell} \cdot (h - \ell) = \mathcal{O}(2^h) = \mathcal{O}(n)$$

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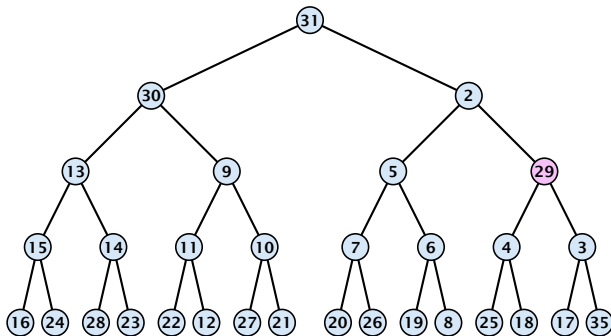
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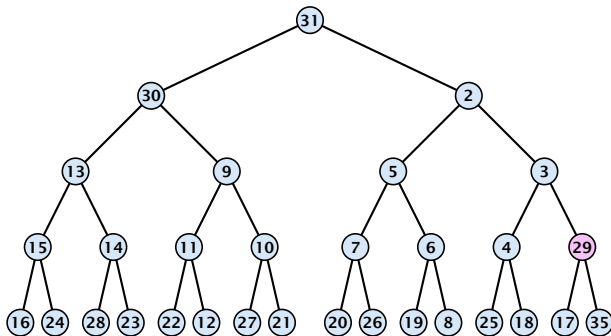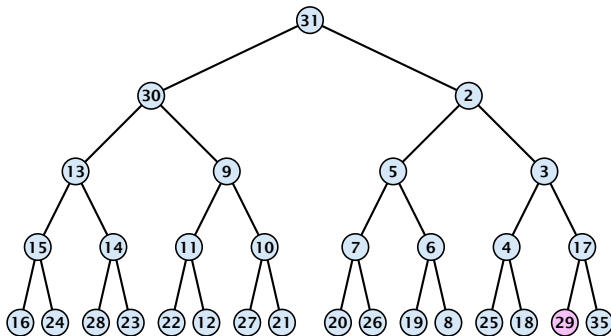
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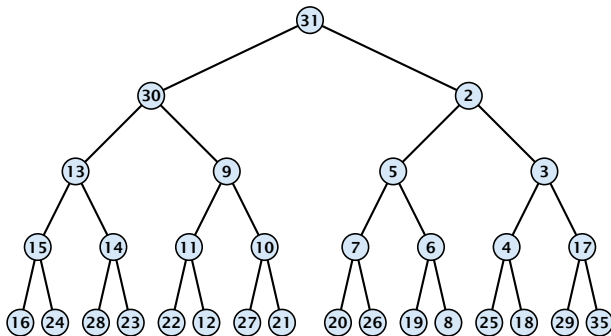
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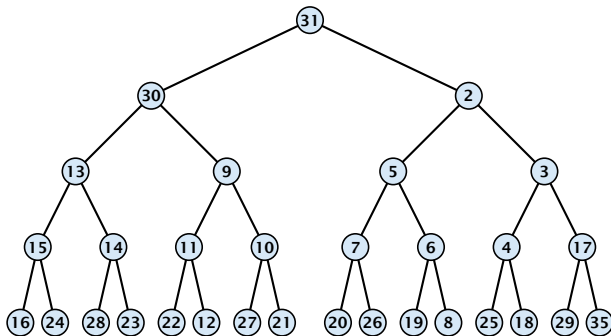We can build a heap in linear time:

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Binary Heaps

Operations:

- **minimum():** Return the root-element. Time $\mathcal{O}(1)$.
- **is-empty():** Check whether root-pointer is null. Time $\mathcal{O}(1)$.
- **insert(k):** Insert at x and bubble up. Time $\mathcal{O}(\log n)$.
- **delete(h):** Swap with x and bubble up or sift-down. Time $\mathcal{O}(\log n)$.
- **build($x_1, \ldots, x_n$):** Insert elements arbitrarily; then do sift-down operations starting with the lowest layer in the tree. Time $\mathcal{O}(n)$.

Binary Heaps

The standard implementation of binary heaps is via arrays. Let $A[0, \ldots, n-1]$ be an array

- The parent of i-th element is at position $\lfloor \frac{i-1}{2} \rfloor$.
- The left child of i-th element is at position $2i + 1$.
- The right child of i-th element is at position $2i + 2$.

Finding the successor of x is much easier than in the description on the previous slide. Simply increase or decrease x.

The resulting binary heap is not addressable. The elements don't maintain their positions and therefore there are no stable handles.

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8.2 Binomial Heaps

Operation	Binary Heap	BST	Binomial Heap	Fibonacci Heap*
build	n	$n \log n$	$n \log n$	n
minimum	1	$\log n$	$\log n$	1
is-empty	1	1	1	1
insert	$\log n$	$\log n$	$\log n$	1
delete	$\log n^{**}$	$\log n$	$\log n$	$\log n$
delete-min	$\log n$	$\log n$	$\log n$	$\log n$
decrease-key	$\log n$	$\log n$	$\log n$	1
merge	n	$n \log n$	$\log n$	1

Binomial Trees

Binomial Trees

Properties of Binomial Trees

- B_k has 2^k nodes.
- B_k has height k.
- The root of B_k has degree k.
- B_k has $\binom{k}{\ell}$ nodes on level ℓ.
- Deleting the root of B_k gives trees $B_0, B_1, \ldots, B_{k-1}$.

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Binomial Trees

Deleting the root of B_5 leaves sub-trees B_4, B_3, B_2, B_1, and B_0.

Deleting the leaf furthest from the root (in B_5) leaves a path that connects the roots of sub-trees B_4, B_3, B_2, B_1, and B_0.

Binomial Trees
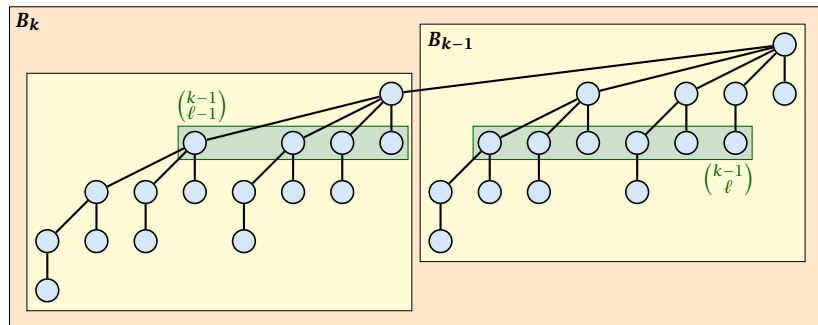
The number of nodes on level ℓ in tree B_k is therefore

$$\binom{k-1}{\ell-1} + \binom{k-1}{\ell} = \binom{k}{\ell}$$

Binomial Trees

The binomial tree B_k is a sub-graph of the hypercube H_k.

The parent of a node with label $b_n, \ldots, b_1, b_0$ is obtained by setting the least significant 1-bit to 0.

The ℓ-th level contains nodes that have ℓ 1's in their label.

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8.2 Binomial Heaps

How do we implement trees with non-constant degree?

▶ The children of a node are arranged in a circular linked list.

▶ A child-pointer points to an arbitrary node within the list.

▶ A parent-pointer points to the parent node.

▶ Pointers $x.\text{left}$ and $x.\text{right}$ point to the left and right sibling of x (if x does not have siblings then $x.\text{left} = x.\text{right} = x$).

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8.2 Binomial Heaps

▶ Given a pointer to a node x we can splice out the sub-tree rooted at x in constant time.

▶ We can add a child-tree T to a node x in constant time if we are given a pointer to x and a pointer to the root of T.

Binomial Heap

In a binomial heap the keys are arranged in a collection of binomial trees.

Every tree fulfills the heap-property

There is at most one tree for every dimension/order. For example the above heap contains trees B_0, B_1, and B_4.

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Binomial Heap: Merge

Given the number n of keys to be stored in a binomial heap we can deduce the binomial trees that will be contained in the collection.

Let B_{k_1}, B_{k_2}, B_{k_3}, $k_i < k_{i+1}$ denote the binomial trees in the collection and recall that every tree may be contained at most once.

Then $n = \sum_i 2^{k_i}$ must hold. But since the k_i are all distinct this means that the k_i define the non-zero bit-positions in the binary representation of n.

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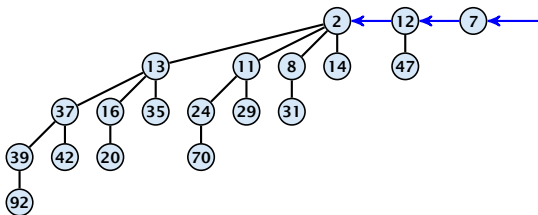
▶ Let $n = b_d b_{d-1}, \ldots, b_0$ denote binary representation of n.

▶ The heap contains tree B_i iff $b_i = 1$.

▶ Hence, at most $\lfloor \log n \rfloor + 1$ trees.

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The merge-operation is instrumental for binomial heaps.

A merge is easy if we have two heaps with different binomial trees. We can simply merge the tree-lists.
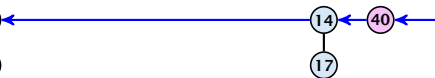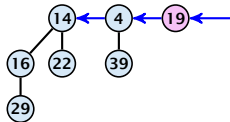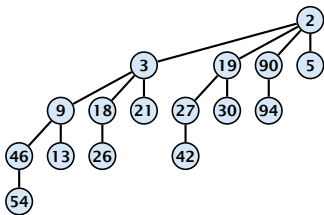
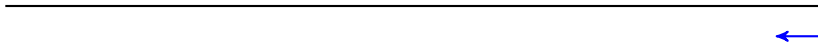Otherwise, we cannot do this because the merged heap is not allowed to contain two trees of the same order.

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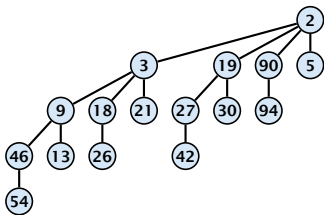
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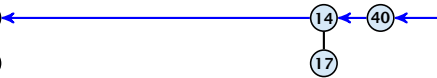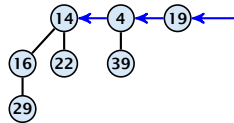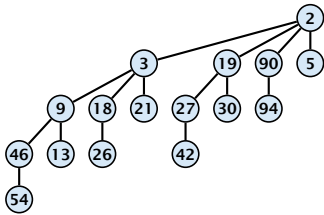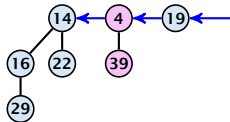For more trees the technique is analogous to binary addition.

Binomial Heap: Merge

The merge-operation is instrumental for binomial heaps.

A merge is easy if we have two heaps with different binomial trees. We can simply merge the tree-lists.

Otherwise, we cannot do this because the merged heap is not allowed to contain two trees of the same order.

Merging two trees of the same size: Add the tree with larger root-value as a child to the other tree.

For more trees the technique is analogous to binary addition.

8.2 Binomial Heaps

$S_1.\,\text{merge}(S_2)$:

- ▶ Analogous to binary addition.
- ▶ Time is proportional to the number of trees in both heaps.
- ▶ Time: $\mathcal{O}(\log n)$.

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8.2 Binomial Heaps

All other operations can be reduced to merge()**.**

$S.$ insert(x):

▶ Create a new heap S' that contains just the element x.

▶ Execute $S.$ merge(S').

▶ Time: $\mathcal{O}(\log n)$.

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All other operations can be reduced to merge().

$S.\,\text{insert}(x)$:

▸ Create a new heap S' that contains just the element x.

▸ Execute $S.\,\text{merge}(S')$.

▸ Time: $\mathcal{O}(\log n)$.

8.2 Binomial Heaps

S. minimum():

- ▶ Find the minimum key-value among all roots.
- ▶ Time: $\mathcal{O}(\log n)$.

8.2 Binomial Heaps

S. delete-min():

- Find the minimum key-value among all roots.

- Remove the corresponding tree $T_{\min}$ from the heap.

- Create a new heap S' that contains the trees obtained from $T_{\min}$ after deleting the root (note that these are just $\mathcal{O}(\log n)$ trees).

- Compute S. merge(S').

- Time: $\mathcal{O}(\log n)$.

8.2 Binomial Heaps

S. **delete-min():**

- ▶ Find the minimum key-value among all roots.
- ▶ Remove the corresponding tree T_{min} from the heap.
- ▶ Create a new heap S' that contains the trees obtained from T_{min} after deleting the root (note that these are just $\mathcal{O}(\log n)$ trees).
- ▶ Compute S. merge(S').
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- ▶ Compute $S.\mathrm{merge}(S')$.
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8.2 Binomial Heaps

S. decrease-key(handle h):

▶ Decrease the key of the element pointed to by h.

▶ Bubble the element up in the tree until the heap property is fulfilled.

▶ Time: $\mathcal{O}(\log n)$ since the trees have height $\mathcal{O}(\log n)$.

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8.2 Binomial Heaps

S. delete(handle h):

- Execute S. decrease-key$(h, -\infty)$.
- Execute S. delete-min().
- Time: $\mathcal{O}(\log n)$.

S. delete(handle h):

- Execute $S.\text{decrease-key}(h, -\infty)$.
- Execute $S.\text{delete-min}()$.
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8.2 Binomial Heaps

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- ▸ Execute S. decrease-key$(h, -\infty)$.
- ▸ Execute S. delete-min$()$.
- ▸ Time: $\mathcal{O}(\log n)$.

8.3 Fibonacci Heaps

Collection of trees that fulfill the heap property.

Structure is much more relaxed than binomial heaps.
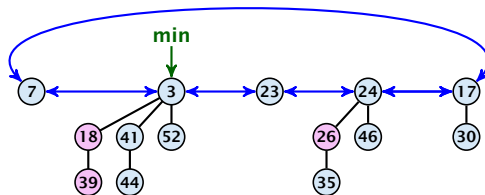
8.3 Fibonacci Heaps

Additional implementation details:

- Every node x stores its degree in a field $x.\,\mathrm{degree}$. Note that this can be updated in constant time when adding a child to x.

- Every node stores a boolean value $x.\,\mathrm{marked}$ that specifies whether x is marked or not.

8.3 Fibonacci Heaps

The potential function:

- ▶ $t(S)$ denotes the number of trees in the heap.
- ▶ $m(S)$ denotes the number of marked nodes.
- ▶ We use the potential function $\Phi(S) = t(S) + 2m(S)$.
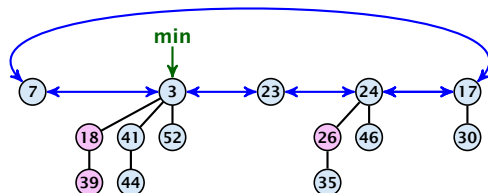
The potential is $\Phi(S) = 5 + 2 \cdot 3 = 11$.

8.3 Fibonacci Heaps

We assume that one unit of potential can pay for a constant amount of work, where the constant is chosen "big enough" (to take care of the constants that occur).

To make this more explicit we use c to denote the amount of work that a unit of potential can pay for.

8.3 Fibonacci Heaps

$S.$ **minimum()**

- ▶ Access through the min-pointer.
- ▶ Actual cost $\mathcal{O}(1)$.
- ▶ No change in potential.
- ▶ Amortized cost $\mathcal{O}(1)$.

8.3 Fibonacci Heaps
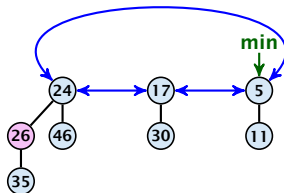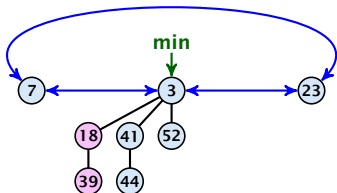
$S.\,\mathbf{merge}(S')$

- ▶ Merge the root lists.
- ▶ Adjust the min-pointer

8.3 Fibonacci Heaps

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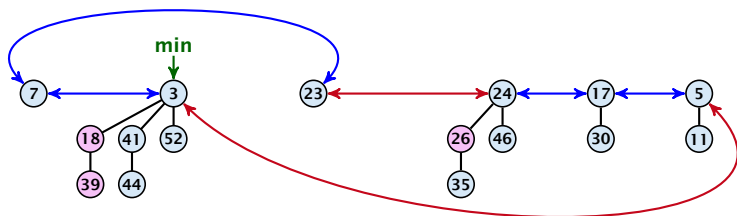
Running time:

- ▶ Actual cost $\mathcal{O}(1)$.

8.3 Fibonacci Heaps

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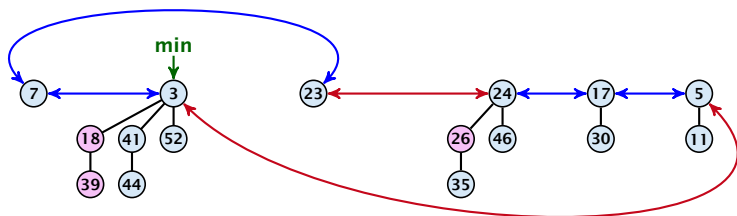
Running time:

- ▶ Actual cost $\mathcal{O}(1)$.
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8.3 Fibonacci Heaps

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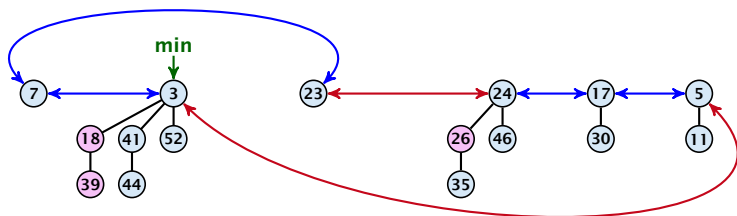
Running time:

- ▶ Actual cost $\mathcal{O}(1)$.
- ▶ No change in potential.
- ▶ Hence, amortized cost is $\mathcal{O}(1)$.

8.3 Fibonacci Heaps

S. **insert**(*x*)

- ▶ Create a new tree containing *x*.
- ▶ Insert *x* into the root-list.
- ▶ Update min-pointer, if necessary.
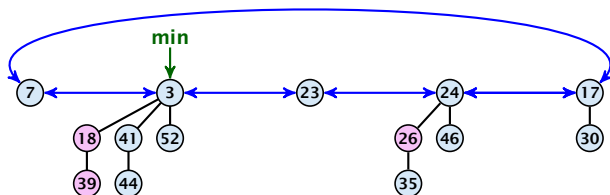
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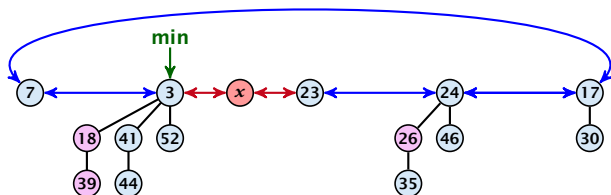
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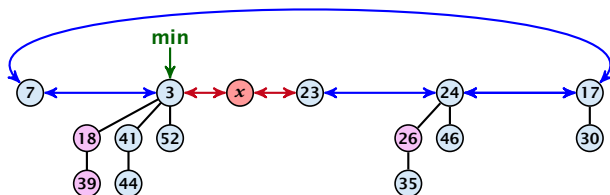
Running time:

- ▶ Actual cost $\mathcal{O}(1)$.
- ▶ Change in potential is $+1$.
- ▶ Amortized cost is $c + \mathcal{O}(1) = \mathcal{O}(1)$.
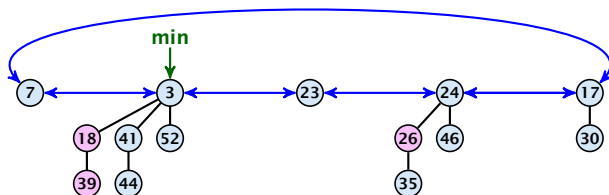
S. delete-min(x)

8.3 Fibonacci Heaps
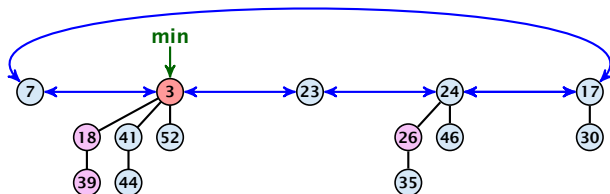
S. **delete-min(*x*)**

▶ Delete minimum; add child-trees to heap;
  time: $D(\min) \cdot \mathcal{O}(1)$.

8.3 Fibonacci Heaps

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time: $D(\min) \cdot \mathcal{O}(1)$.

▶ Update min-pointer; time: $(t + D(\min)) \cdot \mathcal{O}(1)$.
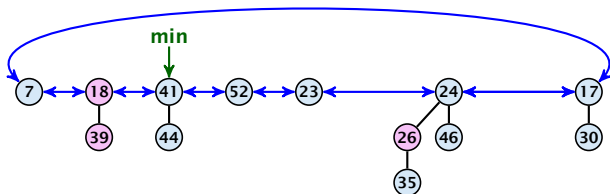
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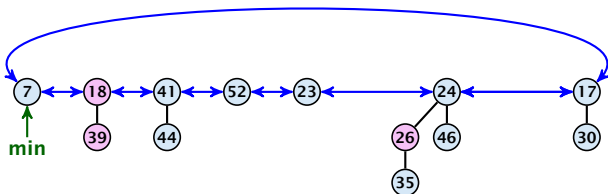
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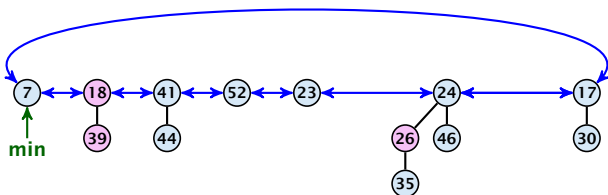
- ▶ Consolidate root-list so that no roots have the same degree.
  Time $t \cdot \mathcal{O}(1)$ (see next slide).

8.3 Fibonacci Heaps

Consolidate:

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8.3 Fibonacci Heaps

Actual cost for delete-min()

▶ At most $D_n + t$ elements in root-list before consolidate.

8.3 Fibonacci Heaps

Actual cost for delete-min()

- ▶ At most $D_n + t$ elements in root-list before consolidate.
- ▶ Actual cost for a delete-min is at most $\mathcal{O}(1) \cdot (D_n + t)$.
  Hence, there exists c_1 s.t. actual cost is at most $c_1 \cdot (D_n + t)$.

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▶ $t' \leq D_n + 1$ as degrees are different after consolidating.

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- ▶ $t' \leq D_n + 1$ as degrees are different after consolidating.
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Amortized cost for delete-min()

▶ $t' \leq D_n + 1$ as degrees are different after consolidating.

▶ Therefore $\Delta\Phi \leq D_n + 1 - t$;

▶ We can pay $c \cdot (t - D_n - 1)$ from the potential decrease.

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$$c_1 \cdot (D_n + t) - c \cdot (t - D_n - 1)$$
$$\le (c_1 + c)D_n + (c_1 - c)t + c$$

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$$\leq (c_1 + c)D_n + (c_1 - c)t + c \leq 2c(D_n + 1)$$

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for $c \geq c_1$.

8.3 Fibonacci Heaps

If the input trees of the consolidation procedure are binomial trees (for example only singleton vertices) then the output will be a set of distinct binomial trees, and, hence, the Fibonacci heap will be (more or less) a Binomial heap right after the consolidation.

If we do not have delete or decrease-key operations then $D_n \leq \log n$.

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Case 1: decrease-key does not violate heap-property

▶ Just decrease the key-value of element referenced by h. Nothing else to do.

Fibonacci Heaps: decrease-key(handle h, v)

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Fibonacci Heaps: decrease-key(handle h, v)

Case 2: heap-property is violated, but parent is not marked

▶ Decrease key-value of element x reference by h.

▶ If the heap-property is violated, cut the parent edge of x, and make x into a root.

▶ Adjust min-pointers, if necessary.

▶ Mark the (previous) parent of x (unless it's a root).

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8.3 Fibonacci Heaps

Fibonacci Heaps: decrease-key(handle h, v)

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Fibonacci Heaps: decrease-key(handle h, v)

Case 3: heap-property is violated, and parent is marked

▶ Decrease key-value of element x reference by h.

▶ Cut the parent edge of x, and make x into a root.

▶ Adjust min-pointers, if necessary.

▶ Continue cutting the parent until you arrive at an unmarked node.

Fibonacci Heaps: decrease-key(handle h, v)

Case 3: heap-property is violated, and parent is marked

▸ Decrease key-value of element x reference by h.

▸ Cut the parent edge of x, and make x into a root.

▸ Adjust min-pointers, if necessary.

▸ Continue cutting the parent until you arrive at an unmarked node.

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▶ Cut the parent edge of x, and make x into a root.

▶ Adjust min-pointers, if necessary.

▶ Execute the following:

> $p \leftarrow \text{parent}[x]$;
> while (p is marked)
> > $pp \leftarrow \text{parent}[p]$;
> > cut of p; make it into a root; unmark it;
> > $p \leftarrow pp$;
>
> if p is unmarked and not a root mark it;

Fibonacci Heaps: decrease-key(handle h, v)

Actual cost:

- Constant cost for decreasing the value.
- Constant cost for each of ℓ cuts.
- Hence, cost is at most $c_2 \cdot (\ell + 1)$, for some constant c_2.

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Amortized cost:

- ▶ $t' = t + \ell$, as every cut creates one new root.
- ▶ $m' \le m - (\ell - 1) + 1 = m - \ell + 2$, since all but the first cut unmarks a node; the last cut may mark a node.
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$$c_2(\ell + 1) + c(4 - \ell) \leq (c_2 - c)\ell + 4c + c_2 = \mathcal{O}(1),$$

$$\text{if } c \geq c_2.$$

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Delete node

$H.\,\mathbf{delete}(x)$:

- ▶ decrease value of x to $-\infty$.
- ▶ delete-min.

Amortized cost: $\mathcal{O}(D_n)$

- ▶ $\mathcal{O}(1)$ for decrease-key.
- ▶ $\mathcal{O}(D_n)$ for delete-min.

8.3 Fibonacci Heaps

Lemma 24

Let x be a node with degree k and let $y_1, \ldots, y_k$ denote the children of x in the order that they were linked to x. Then

$$\operatorname{degree}(y_i) \geq \begin{cases} 0 & \textit{if } i = 1 \\ i - 2 & \textit{if } i > 1 \end{cases}$$

8.3 Fibonacci Heaps

Proof

▶ When y_i was linked to x, at least $y_1, \ldots, y_{i-1}$ were already linked to x.

▶ Hence, at this time $\mathrm{degree}(x) \geq i - 1$, and therefore also $\mathrm{degree}(y_i) \geq i - 1$ as the algorithm links nodes of equal degree only.

▶ Since, then y_i has lost at most one child.

▶ Therefore, $\mathrm{degree}(y_i) \geq i - 2$.

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8.3 Fibonacci Heaps

▶ Let s_k be the minimum possible size of a sub-tree rooted at a node of degree k that can occur in a Fibonacci heap.

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Let x be a degree k node of size s_k and let $y_1, \ldots, y_k$ be its children.

$$s_k = 2 + \sum_{i=2}^{k} \text{size}(y_i)$$

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$$= 2 + \sum_{i=0}^{k-2} s_i$$

8.3 Fibonacci Heaps

Definition 25

Consider the following non-standard Fibonacci type sequence:

$$F_k = \begin{cases} 1 & \text{if } k = 0 \\ 2 & \text{if } k = 1 \\ F_{k-1} + F_{k-2} & \text{if } k \geq 2 \end{cases}$$

Facts:

1. $F_k \geq \phi^k$.
2. For $k \geq 2$: $F_k = 2 + \sum_{i=0}^{k-2} F_i$.

The above facts can be easily proved by induction. From this it follows that $s_k \geq F_k \geq \phi^k$, which gives that the maximum degree in a Fibonacci heap is logarithmic.

k=0:	$1 = F_0 \geq \Phi^0 = 1$
k=1:	$2 = F_1 \geq \Phi^1 \approx 1.61$
k-2,k-1→ k:	$F_k = F_{k-1} + F_{k-2} \geq \Phi^{k-1} + \Phi^{k-2} = \Phi^{k-2} \overbrace{(\Phi + 1)}^{\Phi^2} = \Phi^k$

| k=2: | $3 = F_2 = 2 + 1 = 2 + F_0$ |
| k-1→ k: | $F_k = F_{k-1} + F_{k-2} = 2 + \sum_{i=0}^{k-3} F_i + F_{k-2} = 2 + \sum_{i=0}^{k-2} F_i$ |

8.3 Fibonacci Heaps

9 Union Find

Union Find Data Structure $\mathcal{P}$: Maintains a partition of disjoint sets over elements.

- ▶ $\mathcal{P}.\,\mathrm{makeset}(x)$: Given an element x, adds x to the data-structure and creates a singleton set that contains only this element. Returns a locator/handle for x in the data-structure.

- ▶ $\mathcal{P}.\,\mathrm{find}(x)$: Given a handle for an element x; find the set that contains x. Returns a representative/identifier for this set.

- ▶ $\mathcal{P}.\,\mathrm{union}(x, y)$: Given two elements x, and y that are currently in sets S_x and S_y, respectively, the function replaces S_x and S_y by $S_x \cup S_y$ and returns an identifier for the new set.

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Applications:

▶ Keep track of the connected components of a dynamic graph that changes due to insertion of nodes and edges.

▶ Kruskals Minimum Spanning Tree Algorithm

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▶ Kruskals Minimum Spanning Tree Algorithm

Algorithm 1 Kruskal-MST($G = (V, E), w$)

1: $A \leftarrow \emptyset$;
2: **for all** $v \in V$ **do**
3: $\quad v.\text{set} \leftarrow \mathcal{P}.\text{makeset}(v.\text{label})$
4: sort edges in non-decreasing order of weight w
5: **for all** $(u, v) \in E$ in non-decreasing order **do**
6: $\quad$ **if** $\mathcal{P}.\text{find}(u.\text{set}) \neq \mathcal{P}.\text{find}(v.\text{set})$ **then**
7: $\quad\quad A \leftarrow A \cup \{(u, v)\}$
8: $\quad\quad \mathcal{P}.\text{union}(u.\text{set}, v.\text{set})$

List Implementation

▶ The elements of a set are stored in a list; each node has a backward pointer to the head.

▶ The head of the list contains the identifier for the set and a field that stores the size of the set.

▶ makeset(x) can be performed in constant time.

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List Implementation

union(*x*, *y*)

- ▸ Determine sets S_x and S_y.

- ▸ Traverse the smaller list (say S_y), and change all backward pointers to the head of list S_x.

- ▸ Insert list S_y at the head of S_x.

- ▸ Adjust the size-field of list S_x.

- ▸ Time: $\min\{|S_x|, |S_y|\}$.

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Running times:

- find(x): constant
- makeset(x): constant
- union(x, y): $\mathcal{O}(n)$, where n denotes the number of elements contained in the set system.

List Implementation

Lemma 26

The list implementation for the ADT union find fulfills the following amortized time bounds:

- $\operatorname{find}(x)$: $\mathcal{O}(1)$.
- $\operatorname{makeset}(x)$: $\mathcal{O}(\log n)$.
- $\operatorname{union}(x, y)$: $\mathcal{O}(1)$.

The Accounting Method for Amortized Time Bounds

▶ There is a bank account for every element in the data structure.

▶ Initially the balance on all accounts is zero.

▶ Whenever for an operation the amortized time bound exceeds the actual cost, the difference is credited to some bank accounts of elements involved.

▶ Whenever for an operation the actual cost exceeds the amortized time bound, the difference is charged to bank accounts of some of the elements involved.

▶ If we can find a charging scheme that guarantees that balances always stay positive the amortized time bounds are proven.

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▶ If we can find a charging scheme that guarantees that balances always stay positive the amortized time bounds are proven.

▶ For an operation whose actual cost exceeds the amortized cost we charge the **excess** to the elements involved.

▶ In total we will charge at most $\mathcal{O}(\log n)$ to an element (regardless of the request sequence).

▶ For each element a makeset operation occurs as the first operation involving this element.

▶ We inflate the amortized cost of the makeset-operation to $\Theta(\log n)$, i.e., at this point we fill the bank account of the element to $\Theta(\log n)$.

▶ Later operations charge the account but the balance never drops below zero.

List Implementation

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List Implementation

makeset(x): The actual cost is $\mathcal{O}(1)$. Due to the cost inflation the amortized cost is $\mathcal{O}(\log n)$.

find(x): For this operation we define the amortized cost and the actual cost to be the same. Hence, this operation does not change any accounts. Cost: $\mathcal{O}(1)$.

union(x, y):

List Implementation

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An element is charged at most $\lfloor \log_2 n \rfloor$ times, where n is the total number of elements in the set system.

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Whenever an element x is charged the number of elements in x's set doubles. This can happen at most $\lfloor \log n \rfloor$ times. □

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- ▶ Maintain nodes of a set in a tree.
- ▶ The root of the tree is the label of the set.
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▶ Let's say a node v sees node x if v is in x's sub-tree at the time that x becomes a child.

▶ A node v sees at most one node of rank s during the running time of the algorithm.

▶ This holds because the rank-sequence of the roots of the different trees that contain v during the running time of the algorithm is a strictly increasing sequence.

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Theorem 31
Union find with path compression fulfills the following amortized running times:

- $\text{makeset}(x) : \mathcal{O}(\log^*(n))$
- $\text{find}(x) : \mathcal{O}(\log^*(n))$
- $\text{union}(x, y) : \mathcal{O}(\log^*(n))$

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Amortized Analysis

What is the total charge made to nodes?

- ▶ The total charge is at most

$$\sum_g n(g) \cdot \text{tow}(g) \ ,$$

  where $n(g)$ is the number of nodes in group g.

Amortized Analysis

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Hence,

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Hence,

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\sum_g n(g) \, \text{tow}(g) \leq n(0) \, \text{tow}(0) + \sum_{g \geq 1} n(g) \, \text{tow}(g)
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Hence,

$$\sum_g n(g) \, \text{tow}(g) \leq n(0) \, \text{tow}(0) + \sum_{g \geq 1} n(g) \, \text{tow}(g) \leq n \log^*(n)$$

Amortized Analysis

Without loss of generality we can assume that all
makeset-operations occur at the start.

This means if we inflate the cost of makeset to $\log^* n$ and add
this to the node account of v then the balances of all node
accounts will sum up to a positive value (this is sufficient to
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The analysis is not tight. In fact it has been shown that the amortized time for the union-find data structure with path compression is $\mathcal{O}(\alpha(m, n))$, where $\alpha(m, n)$ is the inverse Ackermann function which grows a lot lot slower than $\log^* n$. (Here, we consider the average running time of m operations on at most n elements).

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$$A(x, y) = \begin{cases} y + 1 & \text{if } x = 0 \\ A(x - 1, 1) & \text{if } y = 0 \\ A(x - 1, A(x, y - 1)) & \text{otw.} \end{cases}$$

$$\alpha(m, n) = \min\{i \geq 1 : A(i, \lfloor m/n \rfloor) \geq \log n\}$$

▸ $A(0, y) = y + 1$

▸ $A(1, y) = y + 2$

▸ $A(2, y) = 2y + 3$

▸ $A(3, y) = 2^{y+3} - 3$

▸ $A(4, y) = \underbrace{2^{2^{2^2}}}_{y+3 \text{ times}} - 3$

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