

# 17 Augmenting Paths for Matchings

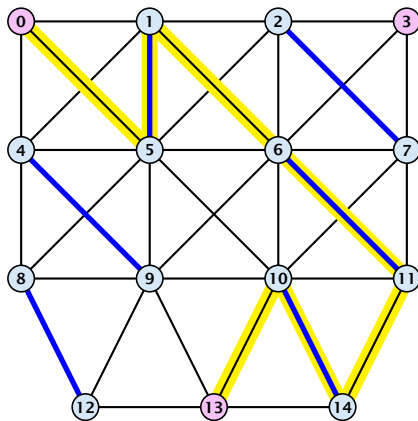
## Definitions.

- ▶ Given a matching  $M$  in a graph  $G$ , a vertex that is not incident to any edge of  $M$  is called a **free vertex** w. r. .t.  $M$ .
- ▶ For a matching  $M$  a path  $P$  in  $G$  is called an **alternating path** if edges in  $M$  alternate with edges not in  $M$ .
- ▶ An alternating path is called an **augmenting path** for matching  $M$  if it ends at distinct free vertices.

## Theorem 1

*A matching  $M$  is a maximum matching if and only if there is no augmenting path w. r. t.  $M$ .*

# Augmenting Paths in Action





## 17 Augmenting Paths for Matchings

### Proof.

- ⇒ If  $M$  is maximum there is no augmenting path  $P$ , because we could switch matching and non-matching edges along  $P$ . This gives matching  $M' = M \oplus P$  with larger cardinality.
- ⇐ Suppose there is a matching  $M'$  with larger cardinality. Consider the graph  $H$  with edge-set  $M' \oplus M$  (i.e., only edges that are in either  $M$  or  $M'$  but not in both).

Each vertex can be incident to at most two edges (one from  $M$  and one from  $M'$ ). Hence, the connected components are alternating cycles or alternating path.

As  $|M'| > |M|$  there is one connected component that is a path  $P$  for which both endpoints are incident to edges from  $M'$ .  $P$  is an augmenting path.

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## Algorithmic idea:

As long as you find an augmenting path augment your matching using this path. When you arrive at a matching for which no augmenting path exists you have a maximum matching.

## Theorem 2

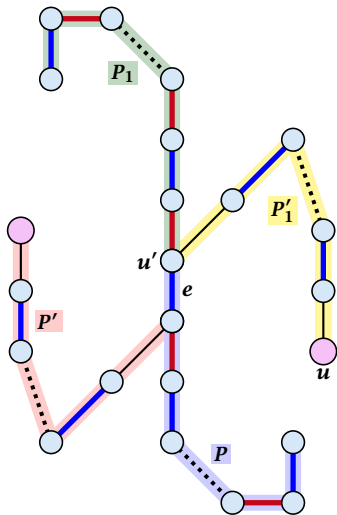
Let  $G$  be a graph,  $M$  a matching in  $G$ , and let  $u$  be a free vertex w.r.t.  $M$ . Further let  $P$  denote an augmenting path w.r.t.  $M$  and let  $M' = M \oplus P$  denote the matching resulting from augmenting  $M$  with  $P$ . If there was no augmenting path starting at  $u$  in  $M$  then there is no augmenting path starting at  $u$  in  $M'$ .

The above theorem allows for an easier implementation of an augmenting path algorithm. Once we checked for augmenting paths starting from  $u$  we don't have to check for such paths in future rounds.

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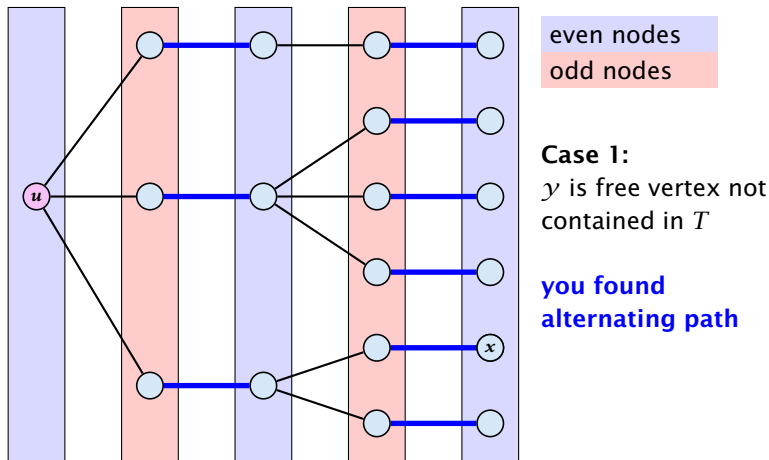
## Proof

- ▶ Assume there is an augmenting path  $P'$  w.r.t.  $M'$  starting at  $u$ .
- ▶ If  $P'$  and  $P$  are node-disjoint,  $P'$  is also augmenting path w.r.t.  $M$  ( $\cancel{f}$ ).
- ▶ Let  $u'$  be the **first** node on  $P'$  that is in  $P$ , and let  $e$  be the matching edge from  $M'$  incident to  $u'$ .
- ▶  $u'$  splits  $P$  into two parts one of which does not contain  $e$ . Call this part  $P_1$ . Denote the sub-path of  $P'$  from  $u$  to  $u'$  with  $P'_1$ .
- ▶  $P_1 \circ P'_1$  is augmenting path in  $M$  ( $\cancel{f}$ ).



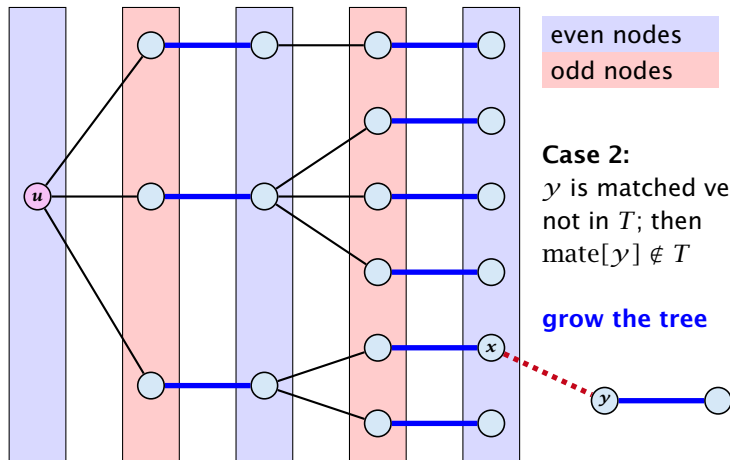
# How to find an augmenting path?

Construct an alternating tree.



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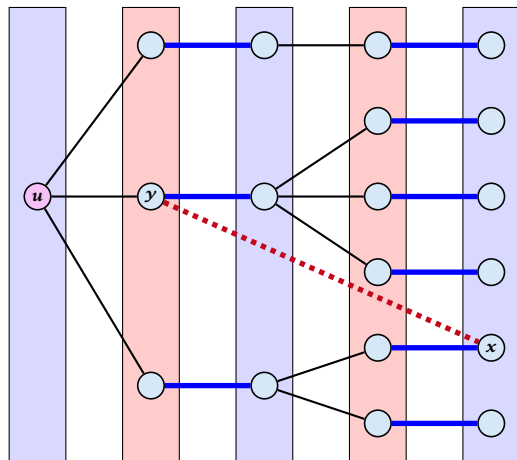
Construct an alternating tree.





# How to find an augmenting path?

Construct an alternating tree.



even nodes

odd nodes

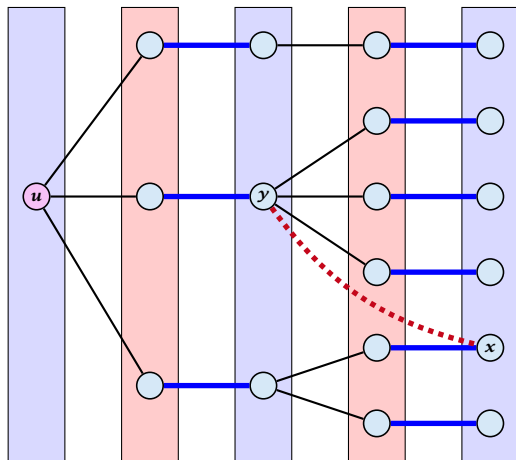
**Case 3:**

$y$  is already contained  
in  $T$  as an odd vertex

**ignore successor  $y$**

# How to find an augmenting path?

Construct an alternating tree.



even nodes

odd nodes

**Case 4:**

$y$  is already contained  
in  $T$  as an even vertex

can't ignore  $y$

does not happen in  
bipartite graphs

### Algorithm 23 BiMatch( $G, match$ )

```
1: for  $x \in V$  do  $mate[x] \leftarrow 0$ ;  
2:  $r \leftarrow 0$ ;  $free \leftarrow n$ ;  
3: while  $free \geq 1$  and  $r < n$  do  
4:    $r \leftarrow r + 1$   
5:   if  $mate[r] = 0$  then  
6:     for  $i = 1$  to  $n$  do  $parent[i'] \leftarrow 0$   
7:      $Q \leftarrow \emptyset$ ;  $Q.append(r)$ ;  $aug \leftarrow false$ ;  
8:     while  $aug = false$  and  $Q \neq \emptyset$  do  
9:        $x \leftarrow Q.dequeue()$ ;  
10:      for  $y \in A_x$  do  
11:        if  $mate[y] = 0$  then  
12:           $augm(mate, parent, y)$ ;  
13:           $aug \leftarrow true$ ;  
14:           $free \leftarrow free - 1$ ;  
15:        else  
16:          if  $parent[y] = 0$  then  
17:             $parent[y] \leftarrow x$ ;  
18:             $Q.enqueue(mate[y])$ ;
```

graph  $G = (S \cup S', E)$

$S = \{1, \dots, n\}$

$S' = \{1', \dots, n'\}$

The lecture version of the slides contains a step-by-step explanation of the algorithm.