

## 6.5 Transformation of the Recurrence

### Example 6

$$f_0 = 1$$

$$f_1 = 2$$

$$f_n = f_{n-1} \cdot f_{n-2} \text{ for } n \geq 2 .$$

Define

$$g_n := \log f_n .$$

Then

$$g_n = g_{n-1} + g_{n-2} \text{ for } n \geq 2$$

$$g_1 = \log 2 = 1 (\text{for } \log = \log_2), \quad g_0 = 0$$

$$g_n = F_n \text{ (} n\text{-th Fibonacci number)}$$

$$f_n = 2^{F_n}$$

## 6.5 Transformation of the Recurrence

### Example 7

$$f_1 = 1$$

$$f_n = 3f_{\frac{n}{2}} + n; \text{ for } n = 2^k, k \geq 1 ;$$

Define

$$g_k := f_{2^k} .$$

Then:

$$g_0 = 1$$

$$g_k = 3g_{k-1} + 2^k, k \geq 1$$

## 6 Recurrences

We get

$$\begin{aligned}g_k &= 3 [g_{k-1}] + 2^k \\&= 3 [3g_{k-2} + 2^{k-1}] + 2^k \\&= 3^2 [g_{k-2}] + 3 \cdot 2^{k-1} + 2^k \\&= 3^2 [3g_{k-3} + 2^{k-2}] + 3 \cdot 2^{k-1} + 2^k \\&= 3^3 g_{k-3} + 3^2 2^{k-2} + 3 \cdot 2^{k-1} + 2^k \\&= 2^k \cdot \sum_{i=0}^k \left(\frac{3}{2}\right)^i \\&= 2^k \cdot \frac{\left(\frac{3}{2}\right)^{k+1} - 1}{1/2} = 3^{k+1} - 2^{k+1}\end{aligned}$$

## 6 Recurrences

Let  $n = 2^k$ :

$$g_k = 3^{k+1} - 2^{k+1}, \text{ hence}$$

$$\begin{aligned} f_n &= 3 \cdot 3^k - 2 \cdot 2^k \\ &= 3(2^{\log_3 3})^k - 2 \cdot 2^k \\ &= 3(2^k)^{\log_3 3} - 2 \cdot 2^k \\ &= 3n^{\log_3 3} - 2n . \end{aligned}$$

# 6 Recurrences

## Bibliography

- [MS08] Kurt Mehlhorn, Peter Sanders:  
*Algorithms and Data Structures — The Basic Toolbox*,  
Springer, 2008
- [CLRS90] Thomas H. Cormen, Charles E. Leiserson, Ron L. Rivest, Clifford Stein:  
*Introduction to algorithms (3rd ed.)*,  
MIT Press and McGraw-Hill, 2009
- [Liu85] Chung Laung Liu:  
*Elements of Discrete Mathematics*  
McGraw-Hill, 1985

The Karatsuba method can be found in [MS08] Chapter 1. Chapter 4.3 of [CLRS90] covers the “Substitution method” which roughly corresponds to “Guessing+induction”. Chapters 4.4, 4.5, 4.6 of this book cover the master theorem. Methods using the characteristic polynomial and generating functions can be found in [Liu85] Chapter 10.