Towards *Terminator 2*:

Self-stabilizing and Distributed Topological Graph Linearization

Stefan Schmid

Joint work with:
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TECHNISCHE UNIVERSITÄT MÜNCHEN

Wroclaw Information Technology Initiative (2008)

Goal: *Terminator 2*!

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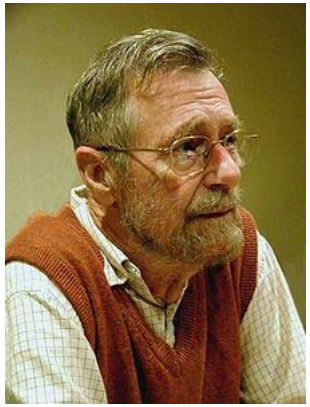
Self-Stabilization (1)

- Important concept in fault-tolerance
- A self-stabilizing system (eventually) ends up in a correct state...
- ... independently of the initial state.

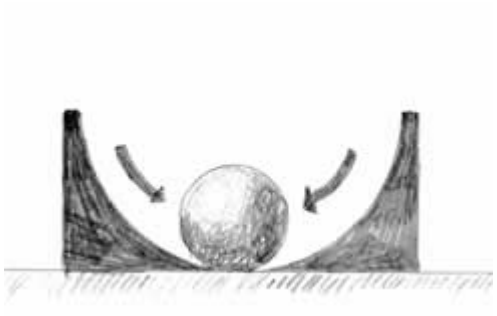


„All the designs I was familiar with were not self-stabilizing in the sense that when once (erroneously) in an illegitimate state, they could – and usually did! – remain so forever.“

E. W. Dijkstra (1974)

Self-Stabilization (2)

- Model: Adversary can disturb the computations (shared variables in system state) arbitrarily
- Once the changes are over, algorithm converges towards desired state



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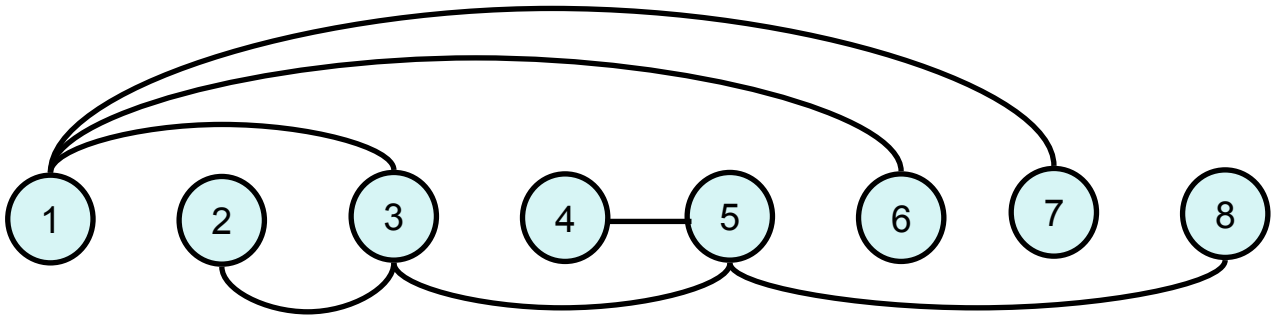
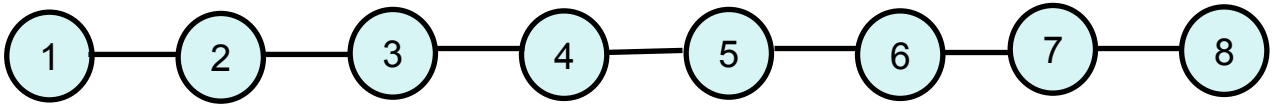
Graph Linearization

- INPUT: Arbitrary connected graph
  - nodes have arbitrary IDs



Graph Linearization

- OUTPUT: Sorted graph



Towards *Terminator 2*:

Self-stabilizing and **Distributed** Topological Graph Linearization

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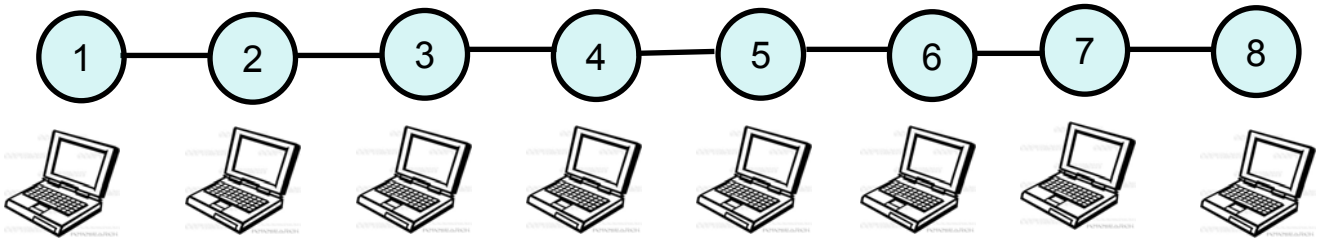
Distributed



- Every node runs its own program:
  - each pair of nodes (u,v) shares a Boolean variable e(u,v) („edge“)
  - program of the node consists of variables and actions
  - an action is of the form:

<name> : <guard> => <commands>

- Guard: predicate over the local and shared variables of node
- Commands: sequence of commands involving any local or shared variables of the node itself or ist neighbors
- An action is enabled if guard is true

Towards *Terminator 2*:

Self-stabilizing and Distributed Topological Graph Linearization

Focus on scalability!

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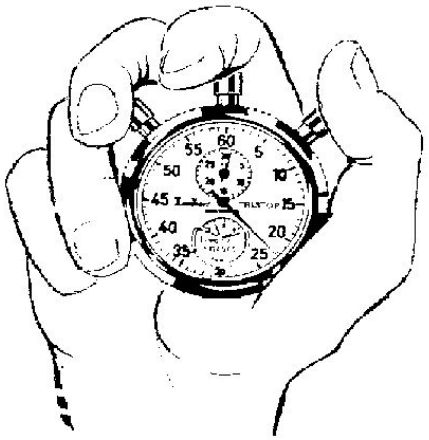
Scalability



- Self-stabilizing algorithm: terminates eventually
- But what about convergence time?
- Analysis of synchronous model
  - total number of rounds (after adversarial change) = execution time
- What can be done in one round?

For scalability reasons, a node should not be involved in too many changes per round!

Talk Outline

1. Two Distributed Algorithms for Graph Linearization
2. Model for Time Complexity of Convergence
3. Analysis and Simulation
4. Conclusion

Two Algorithms

Basic Linearization Step

- A basic linearization step involves a node triple

- Observe: Connectivity is preserved

LIN_{all} and LIN_{max}

- LIN_{all} proposes *all* possible triples to the scheduler (for node *u*)

$$\textbf{linearize left}(v,w): \quad (v,w \in u.L \ \wedge \ w < v < u) \ \rightarrow \ e(u,w) := 0, e(v,w) := 1$$

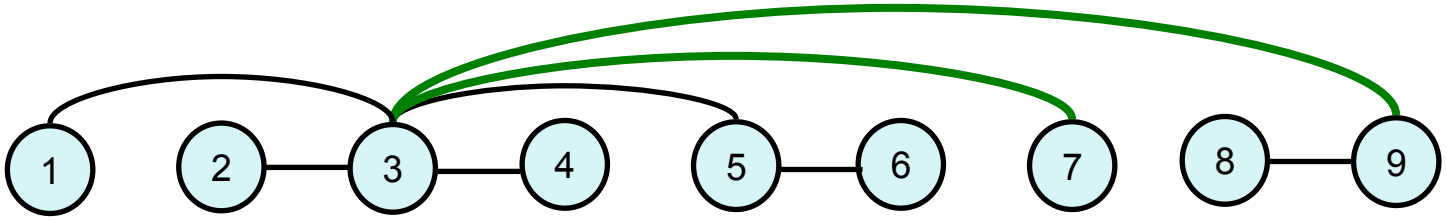
$$\textbf{linearize right}(v,w): \quad (v,w \in u.R \ \wedge \ u < v < w) \ \rightarrow \ e(u,w) := 0, e(v,w) := 1$$

- LIN_{max} proposes the *furthest* triple on each side (for node *u*)

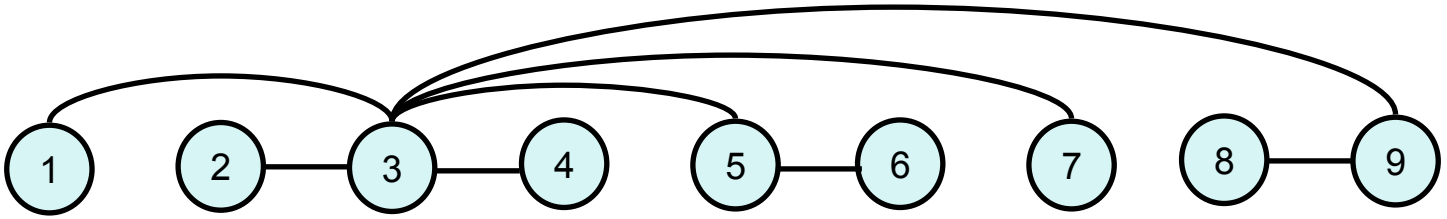
$$\textbf{linearize left}(v,w):$$
$$(v,w \in u.L) \ \wedge \ w < v < u \ \wedge \nexists x \in u.L \setminus \{w\} : x < v) \ \rightarrow \ e(u,w) := 0, e(v,w) := 1$$
$$\textbf{linearize right}(v,w):$$
$$(v,w \in u.R) \ \wedge \ u < v < w \ \wedge \nexists x \in u.R \setminus \{w\} : x > v) \ \rightarrow \ e(u,w) := 0, e(v,w) := 1$$

LIN_{all} and LIN_{max}

- **LIN_{all} proposes *all* possible triples to the scheduler (for node *u*)**



- LIN_{max} proposes the *furthest* triple on each side (for node *u*)



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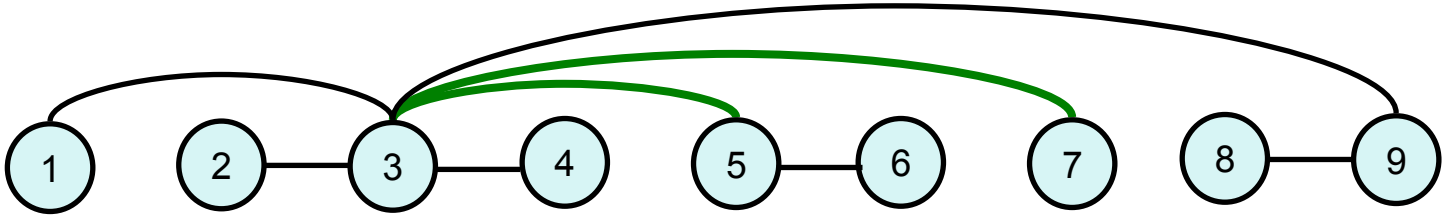

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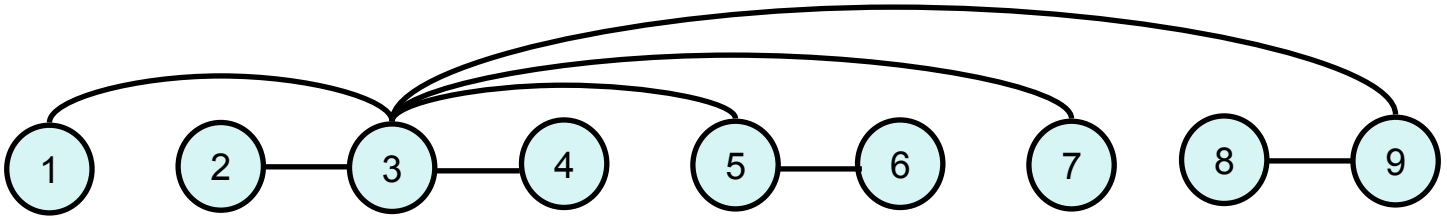

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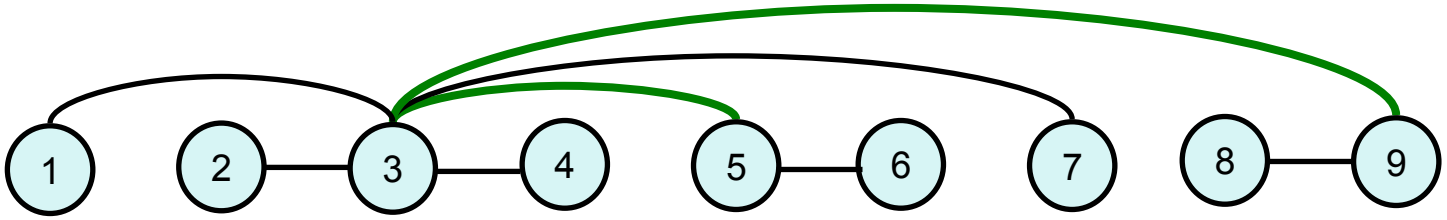

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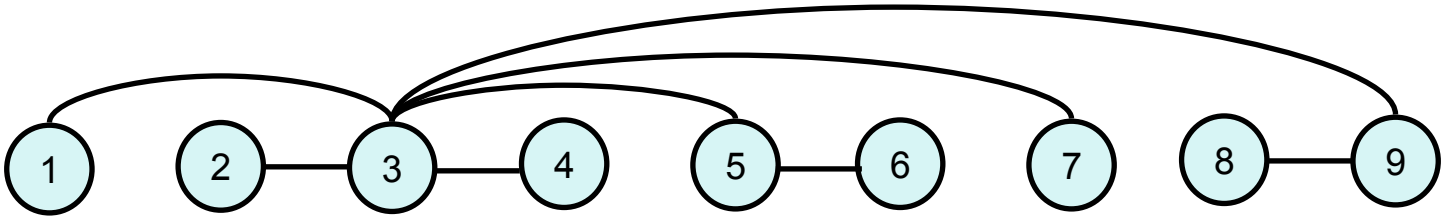

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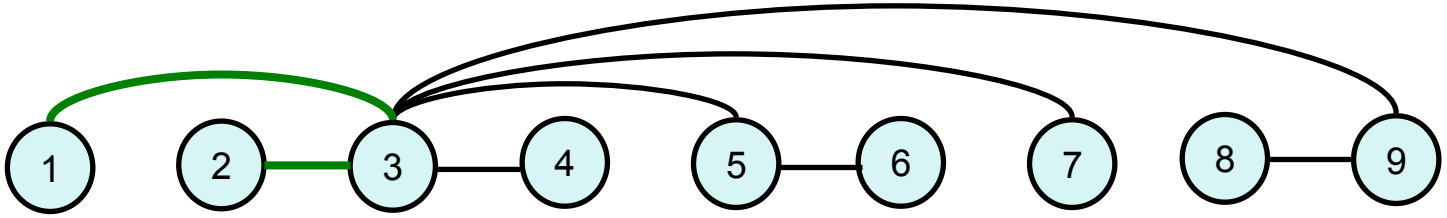

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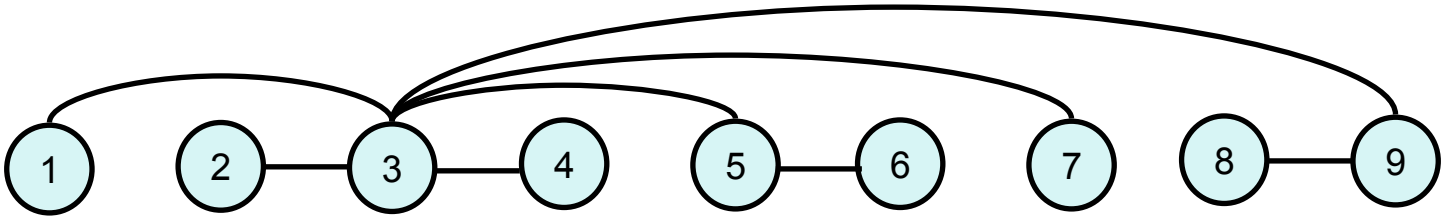

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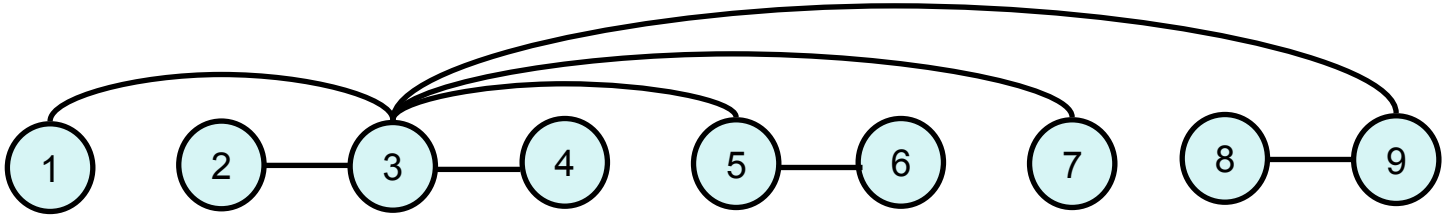

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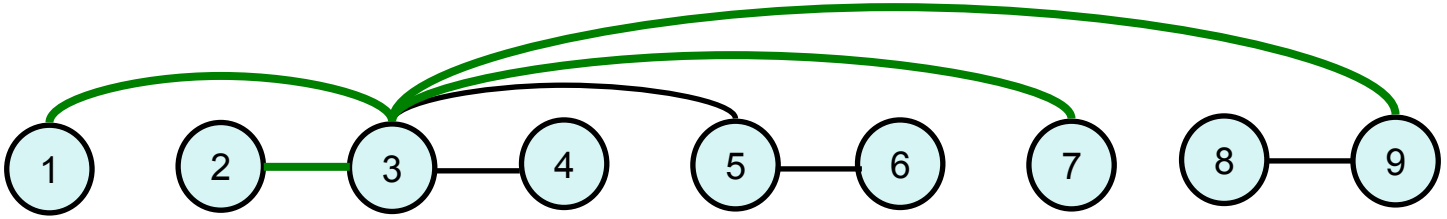
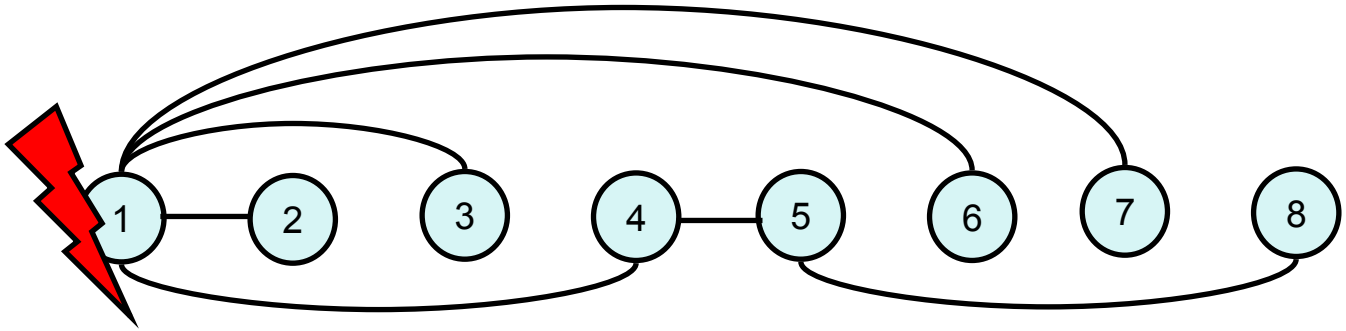


Time Complexity Model

A Naïv Model

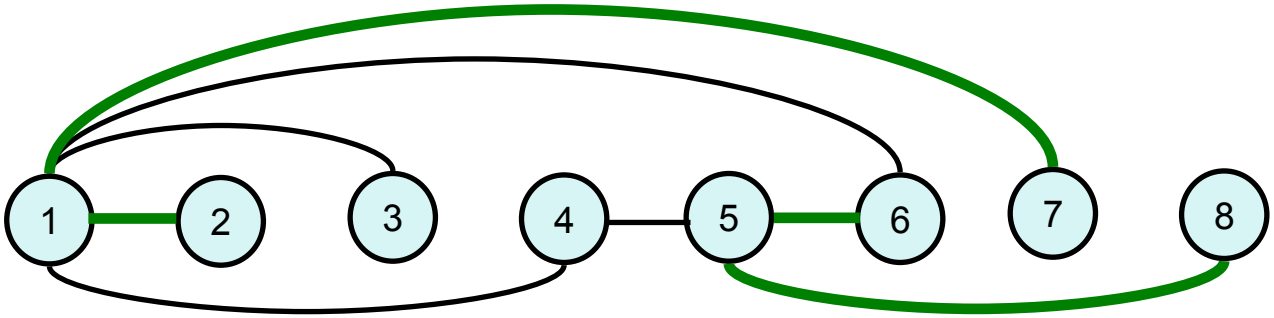
- There are different models for what can happen in one round!
- For example: Every node can fire one action per round
- Problem: Nodes can be involved in many changes
  - Therefore, this solution does not scale!



A Scalable Model

We propose the following, scalable model:

- Let V(A) be the nodes involved in an action A
- Two actions A and B are independent if $V(A) \cap V(B) = \{\}$
- Only an independent set of actions is fired per round



Schedulers

- Nodes propose different enabled actions to the scheduler...
- ... – which one to choose?

Worst-case scheduler: chooses independent set of enabled actions which maximizes the runtime

Best-case scheduler: chooses independent set of enabled actions which minimizes the runtime

Randomized scheduler: chooses independent sets at random

Greedy scheduler: scheduler gives priority to nodes having a large degree

Analysis

Analysis

- It turns out that already these simple algorithms are challenging!
- Overview of results:

Worst-case scheduler:
LIN_{max} requires $\Theta(n^2)$ rounds
LIN_{all} requires $O(n^2 \log n)$ rounds

Greedy scheduler:
LIN_{all} requires $O(n \log n)$ rounds

Best-case scheduler:
LIN_{max} and LIN_{all} require $\Theta(n)$ rounds

With degree cap (worst-case scheduler):
LIN_{max} requires at most $O(n^2)$ and LIN_{all} at most $O(n^3)$ rounds

In Silico Experiments

- In reality, the runtimes are often close to linear (or even constant in „local graphs“ where node i connects to nodes [i-k,i-k+1,...,i-1,i+1,i+2,...i+k])!
- LIN_{all} and LIN_{max} yield a similar performance

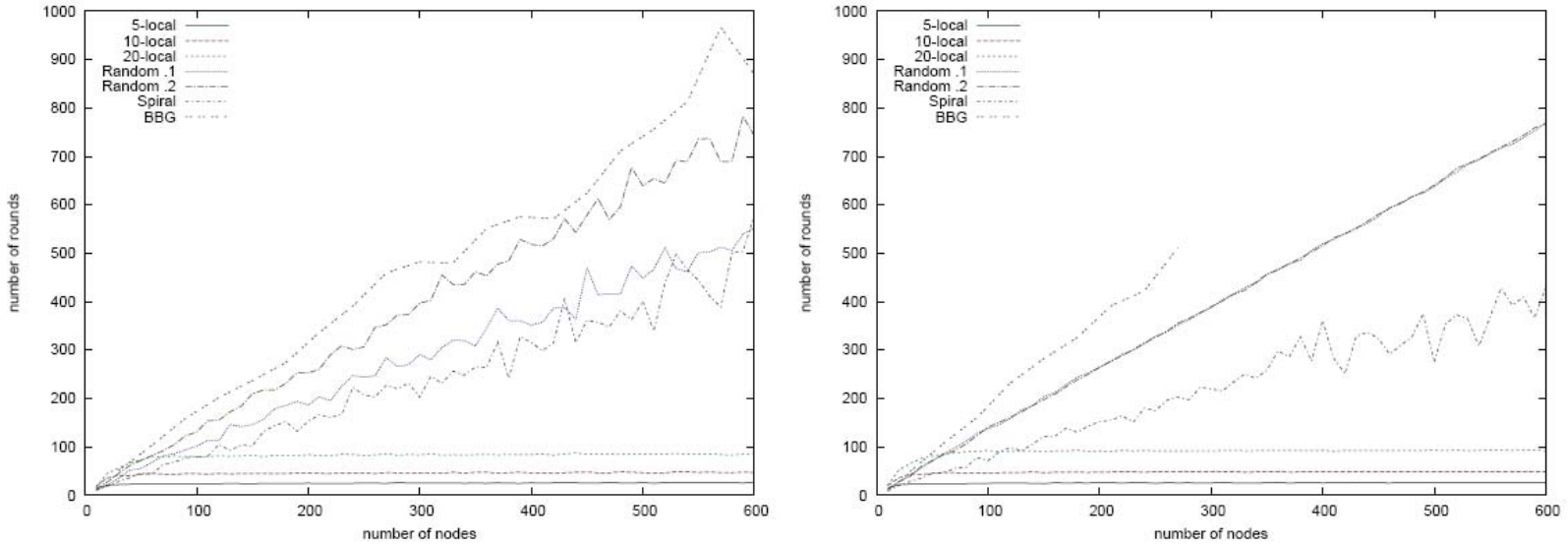


Figure 3: *Left:* Parallel runtime of LIN_{all} for different graphs under $\mathcal{S}_{\text{rand}}$: two k-local graphs with $k = 5$, $k = 10$ and $k = 20$, two random graphs with $p = .1$ and $p = .2$, a spiral graph and a $n/3$-BBG. *Right:* Same experiments with LIN_{max}.

Degree Evolution

- Maximum and average degree do not increase
- Rather, degrees are reduced quickly

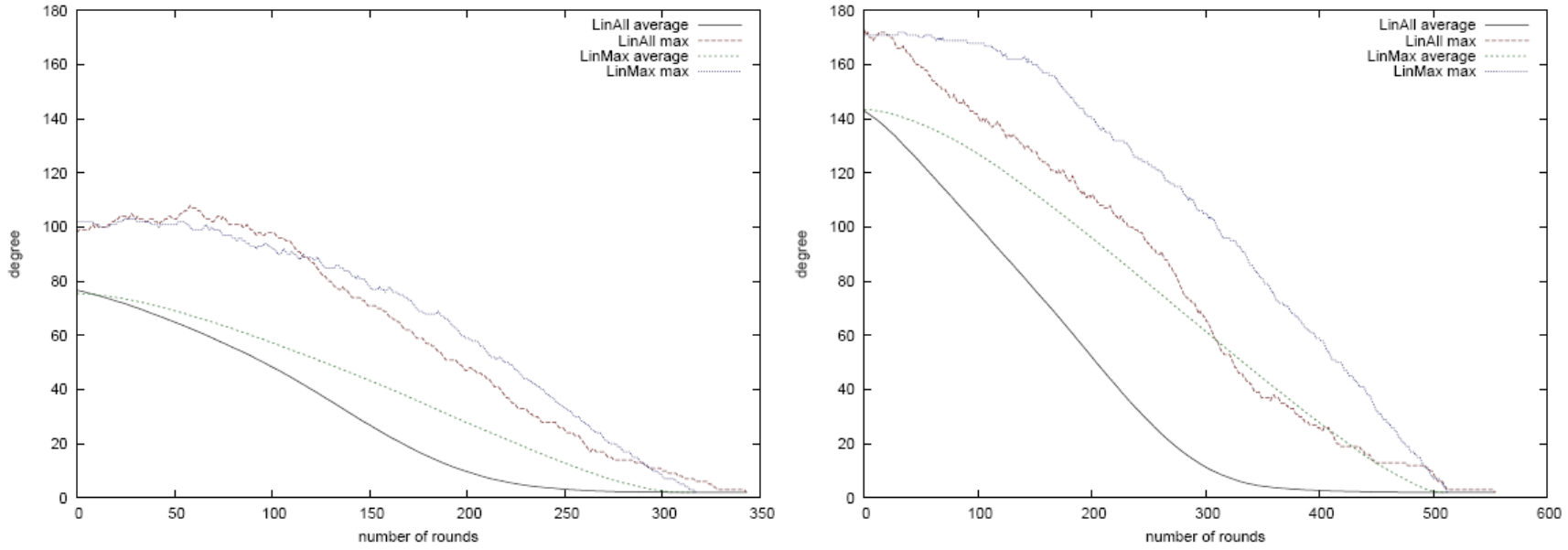
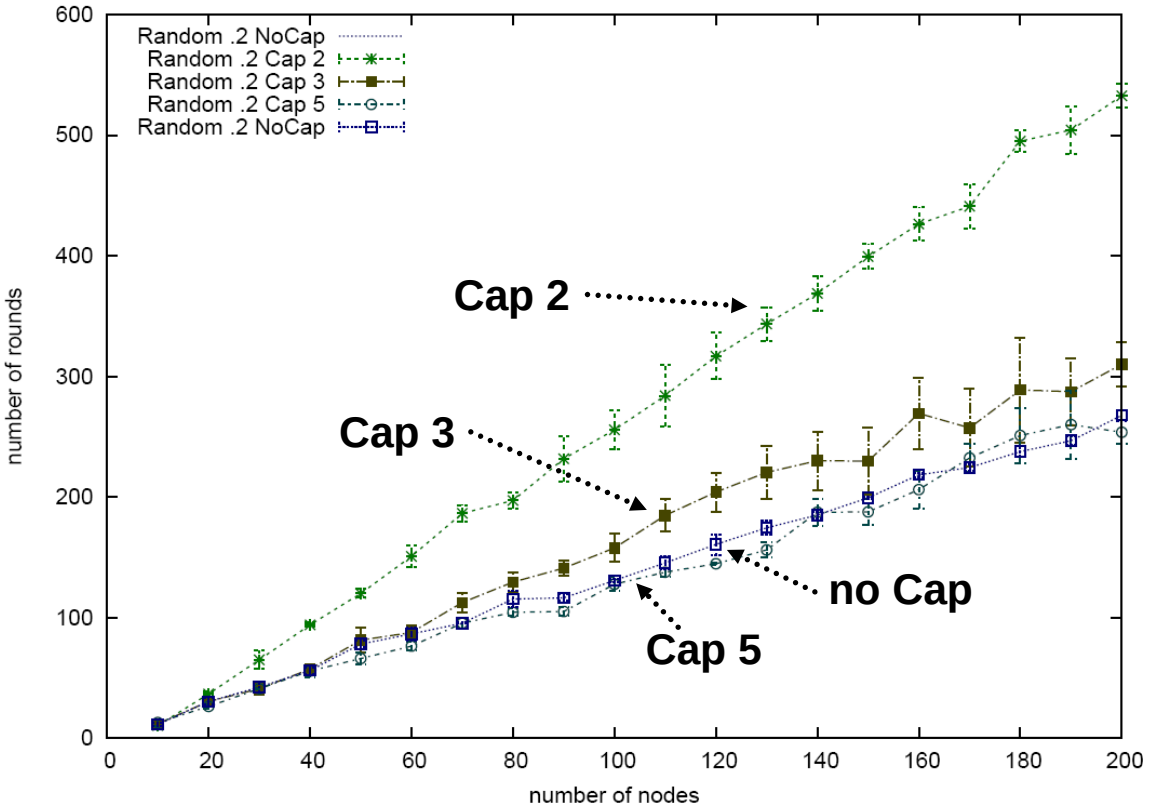


Figure 4: *Left:* Maximum and average degree during a run of LIN_{all} and LIN_{max} on a random graph with edge probability $p = .1$. *Right:* The same experiment on a random graph with $p = .2$.

Degree Cap Phenomenon

- It appears as a degree cap constraint can sometimes improve the runtime!
  - too small degree: blocks many options
  - however, small degree also forces execution on „good paths“



A Sample Analysis (1)

Theorem: Under a worst-case scheduler, LIN_{max} terminates after at most $O(n^2)$ single linearization steps.

Unfortunately, executions can be highly serial and hence the number of linearization steps is asymptotically equivalent to the numer of rounds!

Proof.

Consider the potential function

$$\Phi = \sum_{v \in V} [(2\zeta_l(v) - 1) + (2\zeta_r(v) - 1)] = \sum_{v \in V} 2(\zeta_l(v) + \zeta_r(v) - 1)$$

where ζ(v) is the length of the longest edge out of *v* to the left and right.

A Sample Analysis (2)

$$\Phi = \sum_{v \in V} [(2\zeta_l(v) - 1) + (2\zeta_r(v) - 1)] = \sum_{v \in V} 2(\zeta_l(v) + \zeta_r(v) - 1)$$

Initially $\Phi_0 < 2\,n^2$, as $\zeta_l(v) + \zeta_r(v) < n$ for each node v.

After round i, the potential is at most $\Phi_i < 2\,n^2 - i$.

It most hold for any j that $\Phi_j > 0$, otherwise a node would be isolated.

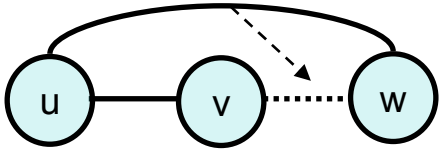
Thus, the claim follows.

A Sample Analysis (3)

$$\Phi = \sum_{v \in V} [(2\zeta_l(v) - 1) + (2\zeta_r(v) - 1)] = \sum_{v \in V} 2(\zeta_l(v) + \zeta_r(v) - 1)$$

Why is $\Phi_i < 2n^2 - i$ true?

Consider a right linearization step:



Case 1: If {*u*,*w*} is also longest edge of *w* to the left.

We remove two longest edges of length |{*u*,*w*}| from Φ.
On the other hand, *u* may have a new longest edge {*u*,*v*} to the right, *v* may have a new longest edge {*v*,*w*} to the right, and *w* a new edge of length at most |{*u*,*w*}|-1 to the left.
Since |{u,w}| = |{u,v}| + |{v,w}|, it follows that

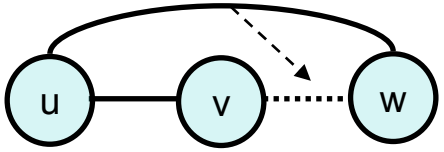
$$\Delta\Phi \leq (2{\cdot}\mathrm{len}(\{u,v\})-1)+(2{\cdot}\mathrm{len}(\{v,w\})-1)+(2(\mathrm{len}(\{u,w\})-1)-1)-(4\cdot\mathrm{len}(\{u,w\})-2) \leq -3$$

A Sample Analysis (4)

$$\Phi = \sum_{v \in V} [(2\zeta_l(v) - 1) + (2\zeta_r(v) - 1)] = \sum_{v \in V} 2(\zeta_l(v) + \zeta_r(v) - 1)$$

Why is $\Phi_i < 2\,n^2 - i$ true?

Consider a right linearization step:



Case 2: If {*u*,*w*} is not longest edge of *w* to the left.

We remove longest edge of length |{*u*,*w*}| from Φ.
On the other hand, *u* may have a new longest edge {*u*,*v*} to the right, *v* may have a new longest edge {*v*,*w*} to the right. In this case

$$\Delta\Phi \leq (2 \cdot \mathrm{len}(\{u, v\}) - 1) + (2 \cdot \mathrm{len}(\{v, w\}) - 1) - (2 \cdot \mathrm{len}(\{u, w\}) - 1) \leq -1$$

QED

Another Sample Analysis (1)

Theorem: Under a greedy scheduler, LIN_{all} terminates after at most O(n log n) rounds.

Greedy scheduler: In each round, nodes are sorted w.r.t. remaining degree (remove fired triples with incident edges). Scheduler picks node *v* with largest degree, and schedules triple of *v* (to the larger degree side) with most distant neighbors.

Proof.

Consider the potential function $\Psi = \sum_{e \in E} \text{len}(e)$

Initially: $\psi_0 < n^3$

In the end: $\psi = n-1$

We will show that in each round, potential ψ is multiplied by a factor of at most 1-1/(24 n). *This implies the claim.*

Another Sample Analysis (2)

This implies the claim?

Lemma 3.4. *Let Ξ be any positive potential function, where Ξ_0 is the initial potential value and Ξ_i is the potential after the i^{th} round of a given algorithm ALG. Assume that $\Xi_i \leq \Xi_{i-1} \cdot (1 - 1/f)$ and that ALG terminates if $\Xi_j \leq \Xi_{stop}$ for some $j \in \mathbb{N}$. Then, the runtime of ALG is at most $O(f \cdot \log(\Xi_0/\Xi_{stop}))$ rounds.*

Proof. From $\Xi_i \leq \Xi_{i-1} \cdot (1 - 1/f)$, it follows that $\Xi_j \leq \Xi_0 \cdot (1 - 1/f)^j$.
Now consider $j = f \cdot \ln \frac{\Xi_0}{\Xi_{stop}}$, which leads to (using $\ln(1 + x) \leq x$ for all $x > -1$)

$$\Xi_j \leq \Xi_0 \cdot (1 - 1/f)^{f \cdot \ln \frac{\Xi_0}{\Xi_{stop}}} = \Xi_0 e^{f \cdot \left(\ln \frac{\Xi_{stop}}{\Xi_0}\right) \cdot \ln(1-1/f)} \leq \Xi_0 e^{f \cdot \left(\ln \frac{\Xi_0}{\Xi_{stop}}\right) \cdot (-1/f)} = \Xi_0 e^{-\ln \frac{\Xi_0}{\Xi_{stop}}} = \Xi_{stop}.$$

□

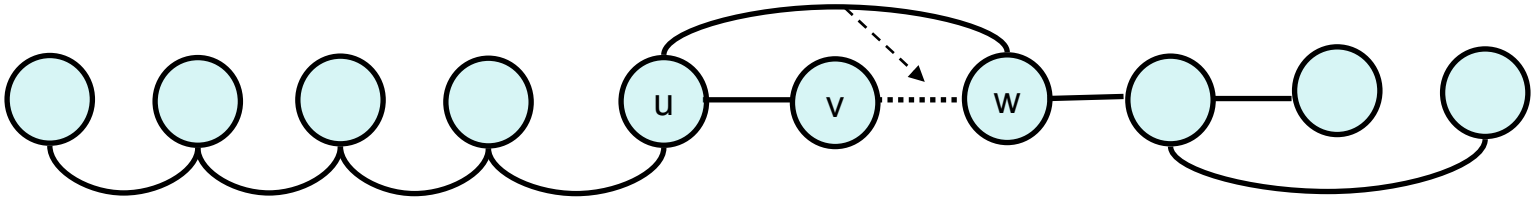
Another Sample Analysis (3)

Greedy scheduler: In each round, nodes are sorted w.r.t. remaining degree (remove fired triples with incident edges). Scheduler picks node *v* with largest degree, and schedules triple of *v* with most distant neighbors (to larger degree side).

Consider the potential function $\Psi = \sum_{e \in E} \text{len}(e)$

We will show that in each round, potential ψ is multiplied by a factor of at most 1-1/(24 n). This implies the claim.

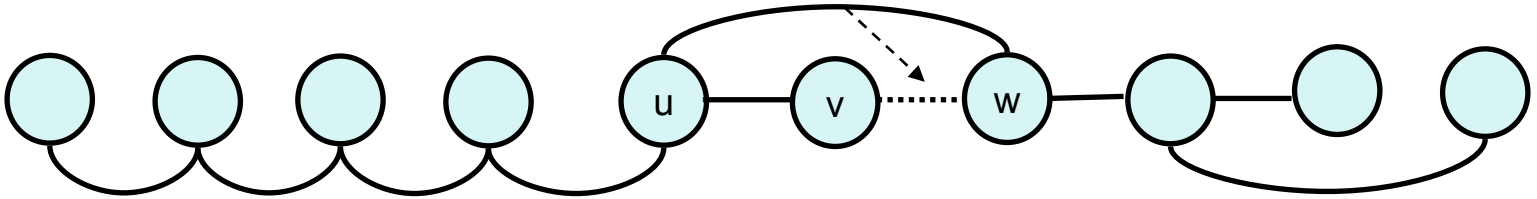
- Observe: firing a triple reduces potential ψ...
- ... but other nodes will be blocked in this round.



- Idea: we want to estimate the amount of blocked potential.

Another Sample Analysis (4)

- Consider the following right-linearization step



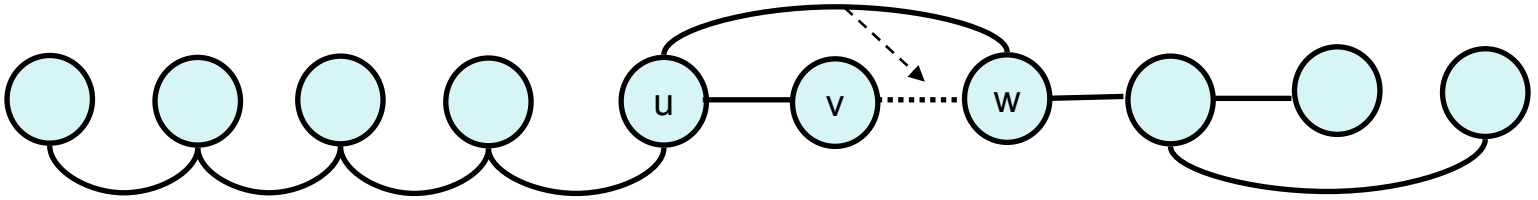
- Removing {u,w} and adding {v,w} reduces the potential by at least

$$dist(u,w)-dist(v,w) = dist(u,v)$$

- Since the greedy scheduler takes larger degree side:

$$dist(u,v) \geq deg(u)/2 - 1 \geq deg(v)/4$$

Another Sample Analysis (5)

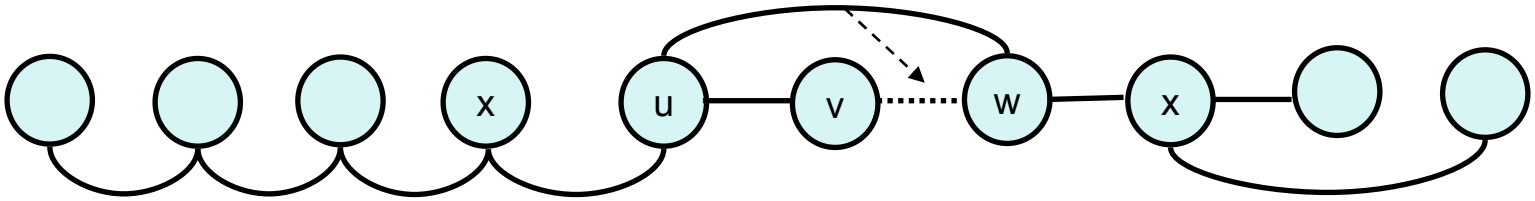


- Thus, potential reduced in one step by at least deg(u)/4
- How much potential is blocked?
- Consider remaining components after removing triple
- Consider neighbor *x* of *u, v* or *w*

  - if x is in ordered line component => blocked potential at most *n+n*

  - if x is in different component => can still be linearized further (account for blocked component's potential later, only count link length potential: *n*)

Another Sample Analysis (6)



- The amount of blocked potential is at most $6 \cdot \deg(u) \cdot n$
  - since *u* has larger degree than *v* and *w*,
  - and since we have at most blocked potential $2 \cdot n$ per neighbor (*n* for component plus *n* for link to this neighbor)

- Thus, potential reduced by a factor at least $1-\Theta(1/n)$ per round.

QED.

Conclusion

Self-stabilizing Graph Linearization

- Most simple algorithms already have many interesting properties
- The quest for faster algorithms has already started!
- Besides linearization, is will be useful to construct alternative graphs in a self-stabilizing manner

Dziekuje!

Slides and papers at
http://www14.informatik.tu-muenchen.de/personen/schmiste/